RETURN THIS COVER SHEET WITH YOUR EXAM AND SOLUTIONS!

Analysis

Ph.D. Preliminary Examination

Department of Mathematics

University of Colorado

January, 2026

- Answer each of the six questions on a separate page. Turn in a page for each problem even if you cannot solve the problem.
- Label each answer sheet with the problem number.
- **Put your number, not your name, in the upper right hand corner of each page**. If you have not received a number, please choose one (1234 for instance) and notify the Graduate Program Assistant (Kellie Geldreich) as to which number you have chosen.

Problem	1	2	3	4	5	6	Total
Points	17	17	17	17	17	17	102
Score							

Problem 1. The *convolution*, denoted $f * g$, of two measurable functions $f, g\colon \mathbb{R} \to \mathbb{R}$ is defined by

$$f * g(x) = \int_{\mathbb{R}} f(x-y)g(y)\,dy,$$

for any $x \in \mathbb{R}$ such that the integral on the right exists (in the Lebesgue sense).

(a) Suppose $f * g(x)$ exists for all $x \in \mathbb{R}$. Show that, if f and g are supported on $(0, \infty)$, then so is $f * g$, and, for $x > 0$,

$$f * g(x) = \int_0^x f(x-y)g(y)\,dy.$$

(b) Show that, if f and g are continuous on $\mathbb{R}$ and supported on $(0, \infty)$, then $f * g$ is also continuous on $\mathbb{R}$ and supported on $(0, \infty)$.

Problem 2. Let $(f_n)_{n=1}^{\infty}$ be a sequence of functions in the space

$$C([0,1]) = \{\text{continuous functions on } [0,1]\}$$

such that, for some constant $c > 0$,

$$||f_n - f_{n+1}|| \leq \frac{c}{n^4}$$

for all natural numbers n. Here, $||\cdot||$ denotes the norm

$$||g|| = \sqrt{\int_0^1 |g(x)|^2\,dx}.$$

(a) Show that $(f_n)_{n=1}^{\infty}$ is a Cauchy sequence with respect to this norm. Hint: for $M > N$, express $f_N - f_M$ as a telescoping sum.

(b) Show that, for any natural number n, the set

$$G_n = \{x \in [0,1] \colon \left|f_n(x) - f_{n+1}(x)\right| \geq n^{-2}\}$$

has measure at most c/n^2. Hint: bound $\int_0^1 \left|f_n(x) - f_{n+1}(x)\right| dx$ below by the integral of the same thing over the subdomain where the integrand is $\geq n^{-2}$.

(c) Show that $(f_n)_{n=1}^{\infty}$ converges pointwise almost everywhere. That is, show that the sequence $(f_n(x))_{n=1}^{\infty}$ converges for almost all $x \in [0,1]$. Hint: given $\varepsilon > 0$, choose a natural number N such that

$$c \sum_{n=N}^{\infty} \frac{1}{n^2} < \varepsilon.$$

Now, for $M > N$ and for

$$x \in S_N = [0,1] - \bigcup_{n=N}^{\infty} G_n,$$

express $f_M(x)$ in terms of $f_N(x)$ and a telescoping sum (see part (a) above). Then take limits as $M \to \infty$.

Problem 3. (The result of this problem is a variant on the *Riemann-Lebesgue Lemma* for $L^1(\mathbb{R})$.)

Define the *Fourier sine transform* $\widehat{f}$ of $f \in L^1(\mathbb{R})$ to be the function on $\mathbb{R}$ defined by

$$\widehat{f}(t) = \int_{\mathbb{R}} f(x) \sin(xt)\, dx.$$

(a) For $t \neq 0$, make a change of variable to show that

$$\widehat{f}(t) = -\int_{\mathbb{R}} f(x + t^{-1}\pi) \sin(xt)\, dx.$$

(b) Use the above two formulas for $\widehat{f}$ to show that

$$\left|\widehat{f}(t)\right| \leq \frac{1}{2} \int_{\mathbb{R}} \left|f(x) - f(x + t^{-1}\pi)\right| dx.$$

(c) Use your above result to show that, if f is continuous and compactly supported on $\mathbb{R}$, then $\widehat{f} \to 0$ as $t \to \pm\infty$.

(d) Extend the result of part (c) above to all $f \in L^1(\mathbb{R})$.

Problem 4.

Let $(X, \mathcal{A}, \mu)$ be a measure space, $A \in \mathcal{A}$ and $f : A \mapsto [-\infty, \infty]$ a μ-integrable function. Show that for any given $\varepsilon > 0$ there is a $\delta > 0$ such that if $B \subset A, B \in \mathcal{A}$ and $\mu(B) < \delta$ then $\int_B |f| d\mu < \varepsilon$.

Problem 5. A normed vector space $(V, \|\cdot\|)$ is called *uniformly convex* if there exists a monotone increasing function $\eta : [0,2] \mapsto [0,1]$ taking positive values on $(0,2]$ such that $\|x\|, \|y\| \leq 1$ implies that

$$\|(x+y)/2\| \leq 1 - \eta(\|x-y\|).$$

Prove that Hilbert spaces are uniformly convex. (Hint: Use the parallelogram rule.)

Problem 6.

Let N denote the set of numbers $x \in \mathbb{R}$ for which there exist infinitely many $p/q \in \mathbb{Q}$ such that

$$|x - (p/q)| < 1/q^3. \tag{1}$$

Prove that N has zero Lebesgue measure. [Hint: Consider the set of numbers $x \in [0,1]$ for which "the denominator q works," and use the Borel-Cantelli Lemma (it states that the summability of the measures of E_i guarantees the set of points belonging to infinitely many E_i's has zero-measure.)]