RETURN THIS COVER SHEET WITH YOUR EXAM AND SOLUTIONS!

Analysis

Ph.D. Preliminary Examination

Department of Mathematics

University of Colorado

January, 2025

- Answer each of the six questions on a separate page. Turn in a page for each problem even if you cannot solve the problem.
- Label each answer sheet with the problem number.
- **Put your number, not your name, in the upper right hand corner of each page**. If you have not received a number, please choose one (1234 for instance) and notify the Graduate Program Assistant (Kellie Geldreich) as to which number you have chosen.

Problem	1	2	3	4	5	6	Total
Points	17	17	17	17	17	17	102
Score							

Problem 1. Let $f, g \in L^1$ be real-valued functions. If

$$\int_E f \leq \int_E g$$

for every measurable set E, prove that $f \leq g$ almost everywhere.

Problem 2.

Let C and α be two *fixed* positive real numbers. Define K to be the set of all real-valued functions on $[0,1]$ satisfying

$$(1) \qquad |f(x) - f(y)| \leq C|x-y|^{\alpha}, \quad x, y \in [0,1].$$

(a) Is K equicontinuous?

Then, with respect to the L^{∞} norm, is K

(b) Closed?

(c) Bounded?

(d) Compact?

Problem 3. Consider two (complex) Hilbert spaces $(H_1, \langle \cdot, \cdot \rangle_1)$ and $(H_2, \langle \cdot, \cdot \rangle_2)$, where $\langle \cdot, \cdot \rangle_j$ denotes the inner product in H_j, j=1, 2. Recall a linear map $F : H_1 \to H_2$ is called unitary if it is bijective and if

$$\langle FX, FY \rangle_2 = \langle X, Y \rangle_1, \quad \text{for all } X, Y \in H_1.$$

(a) Show a linear map F is unitary if and only if F is an isometry and it is surjective. (F is an isometry means that F preserves norms: $\|FX\|_2 = \|X\|_1$).

(b) Show F is a bounded operator and $\|F\| = 1$.

Problem 4. Let $1 < p \leq \infty$. Show that if $f \in L^p(\mathbb{R}^n)$, and

$$\int_{\mathbb{R}^n} fg\,dx = 0$$

for all $g \in C_c^\infty(\mathbb{R}^n)$, then $f = 0$ almost everywhere. (Recall $C_c^\infty(\mathbb{R}^n)$ is the space of all smooth functions with compact support.)

Problem 5. Let (f_n) be a sequence of measurable functions on the finite measure space $(X, \mathcal{F}, \mu)$. If f_n converges to zero almost everywhere and

$$\sup_n \int |f_n|^p \, d\mu < \infty$$

for some $p > 1$, show that

$$\lim_{n\to\infty} \int f_n \, d\mu = 0.$$

[**Hint**: Decompose $\int |f_n| \, d\mu = \int_{|f_n| \le T} |f_n| \, d\mu + \int_{|f_n| > T} |f_n| \, d\mu$ for $T > 0$.]

Problem 6. Suppose that $f \in L^1(\mathbb{R})$ satisfies

$$\lim_{h\to 0} \int_{\mathbb{R}} \left| \frac{f(x+h) - f(x)}{h} \right| \, dm(x) = 0,$$

where m is the Lebesgue measure on $\mathbb{R}$. Show that $f = 0$ almost everywhere.
[**Hint**: Define $F(y) = \int_0^y f(x) \, dm(x)$, and apply the fundamental theorem of calculus for Lebesgue integration to F.]