RETURN THIS COVER SHEET WITH YOUR EXAM AND SOLUTIONS!

Analysis

Ph.D. Preliminary Examination

Department of Mathematics

University of Colorado

January, 2024

- Answer each of the six questions on a separate page. Turn in a page for each problem even if you cannot solve the problem.
- Label each answer sheet with the problem number.
- **Put your number, not your name, in the upper right hand corner of each page**. If you have not received a number, please choose one (1234 for instance) and notify the Graduate Program Assistant (Kellie Geldreich) as to which number you have chosen.

Problem	1	2	3	4	5	6	Total
Points	17	17	17	17	17	17	102
Score							

Problem 1. Determine if the following statement is true or false: If $f : \mathbb{R} \to \mathbb{R}$ satisfies $|f(x) - f(y)| \leq |x - y|$ for all $x, y \in \mathbb{R}$, then f is absolutely continuous. If the statement is true, prove it; if it is false, give a counterexample.

Problem 2. Find a sequence $\{f_n\}$ of continuous functions on $\mathbb{R}$ so that the sequence is uniformly bounded and equicontinuous but does not have a subsequence that converges uniformly on $\mathbb{R}$.

Problem 3. Let m denote the Lebesgue measure on $\mathbb{R}$, and let μ be a measure on the Borel σ-algebra $\mathcal{B}_{\mathbb{R}}$ on $\mathbb{R}$. If $\mu(\mathbb{R}) = 1$, show that

$$\int_{\mathbb{R}} \mu((x, x+a])\, dm(x) = a$$

for any $a > 0$. [**Hint**: Use the Fubini–Tonelli theorem.]

Problem 4.

(a) Let h_n, g_n be measurable real-valued functions such that $h_n \to h$ a.e., and $|h_n| \leq g_n$ for some g_n satisfying $g_n \to g$ a.e., for $g \in L^1$. Show $\int h_n \to \int h$.

[**Hint:** Use Fatou's lemma and rework the proof of the dominated convergence theorem.]

(b) Suppose $f_n, f \in L^1$, and $f_n \to f$ a.e. Prove $\int |f_n - f| \to 0$ if and only if $\int |f_n| \to \int |f|$.

Problem 5. Prove that the smallest constant $c > 0$ for which the inequality

$$\left| \int_0^1 x f(x) dx \right| \leq c \, \|f\|_{\frac{3}{2}}$$

holds for all $f \in L^{\frac{3}{2}}([0,1])$ is $c = (\frac{1}{4})^{\frac{1}{3}}$. Recall, $\|f\|_{\frac{3}{2}} = \{\int |f|^{\frac{3}{2}}\}^{\frac{2}{3}}$.

Problem 6. Let $\{g_n\}$ be a sequence of Lebesgue measurable functions defined on $[0,1]$ so that for some $M < \infty$ and all n, m

- $|g_n(x)| \leq M$ for all $x \in [0,1]$,
- $\int_0^1 g_n(x) g_m(x) dx = \delta_{n,m}$, where $\delta_{n,m}$ is the Kronecker delta: $\delta_{n,m} = 1$ for $n = m$, and $\delta_{n,m} = 0$ otherwise.

Show that for any $f \in L^1([0,1])$,

$$\lim_{n\to\infty} \int_0^1 f(x) g_n(x) dx = 0.$$

[**Hint:** Start with $f \in L^2$.]