RETURN THIS COVER SHEET WITH YOUR EXAM AND SOLUTIONS!

Analysis

Ph.D.
Preliminary Exam

January, 2015

INSTRUCTIONS:

1. Answer each of the six questions on a separate page. Turn in a page for each problem even if you cannot do the problem.

2. Label each answer sheet with the problem number.

3. Put your number, not your name, in the upper right hand corner of each page. If you have not received a number, please choose one (1234 for instance) and notify the graduate secretary as to which number you have chosen.

1. If $g : [0, \infty) \to \mathbb{R}$ is a monotone non-increasing (thus measurable) function satisfying $\lim_{x\to\infty} g(x) = c > 0$, prove that there exists a rational-valued function $h : [0, \infty) \to \mathbb{Q}$ such that the function $f\colon [0, \infty) \to \mathbb{R}$ defined by $f = g \cdot h$ is improperly Riemann integrable on $[0, \infty)$, but not Lebesgue integrable there.

2. Assume that $f\colon [1, 2] \to \mathbb{R}$ is absolutely continuous, with $f(2) = 0$. Prove that

$$\left| \int_1^2 f'(x) \log x \, \mathrm{d}x \right| \le \int_1^2 |f(x)| \, \mathrm{d}x.$$

3. Let $f : [a, b] \to \mathbb{R}$ be a C^1 function. For $\epsilon > 0$, let $C_\epsilon := \{x \in (a, b) : |f'(x)| < \epsilon\}$, and let $A := \{f(x) \mid x \in (a, b), f'(x) = 0\}$.

(i) Prove that C_ϵ is open and that $m(f(C_\epsilon)) < \epsilon \cdot (b - a)$.

(ii) Prove that A has Lebesgue measure zero.

4. Let $(X, \mathcal{B}, \mu)$ be a measure space, and suppose that $p, q, \; r \in (1, \infty)$ satisfy

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1.$$

If $f \in L^p(X, \mu)$, $g \in L^q(X, \mu)$, and $h \in L^r(X, \mu)$, prove that $f \cdot g \cdot h \in L^1(X, \mu)$ and that

$$\|f \cdot g \cdot h\|_1 \le \|f\|_p \cdot \|g\|_q \cdot \|h\|_r.$$

5. Let $(X, \mathcal{B}, \mu)$ be a σ-finite measure space, and suppose that $f : X \to [0, \infty)$ is a nonnegative integrable function. Prove that the function $\psi : [0, \infty) \to [0, \infty]$ defined by $\psi(t) = \mu(\{x \in X : \; f(x) \ge t\})$ is Lebesgue measurable and that

$$\int_X f d\mu = \int_0^\infty \psi(t) \, dt.$$

Hint: you may find Tonelli's Theorem useful.

6. If $\{f_1, f_2, \cdots\}$ is a complete orthonormal set in the Hilbert space $L^2[0, 1]$, where $[0, 1]$ is equipped with Lebesgue measure, and B is an arbitrary measurable subset of positive measure in $[0, 1]$, use Parseval's identity applied to the characteristic function for B to prove that

$$1 \le \int_B \sum_{i=1}^\infty |f_i(x)|^2 dx.$$