RETURN THIS COVER SHEET WITH YOUR EXAM AND SOLUTIONS!

Analysis

Ph.D. Preliminary Exam
Department of Mathematics
University of Colorado Boulder

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INSTRUCTIONS:

1. Answer each of the six questions on a separate page. Turn in a page for each problem even if you cannot do the problem.

2. Label each answer sheet with the problem number.

3. Put your number, not your name, in the upper right hand corner of each page. If you have not received a number, please choose one (1234 for instance) and notify the graduate secretary as to which number you have chosen.

Q.1 a) Let A be a measurable subset of $[0,1]$. Define the function $f\colon [0,1] \to \mathbb{R}$ by setting $f(x) = \mu(A \cap [0,x])$; here μ is the Lebesgue measure. Show that f is absolutely continuous.

b) Does there exist a measurable set $A \subset [0,1]$ such that one has

$$\mu(A \cap [a,b]) = \frac{1}{2}(b-a)$$

for every interval $[a,b] \subset [0,1]$?

Q.2 Let $f \in L^p(\mathbb{R})$, $1 \leq p < \infty$ Set $f_n(x) = f(x + \frac{1}{n})$. Show that the sequence f_n converges to f in L^p. Is this true for $p = \infty$?

Q.3 Let f_n be a sequence of continuous functions on $[0,1]$ such that $|f_n(x)| \leq 1$ for all $n \in \mathbb{N}$, $x \in [0,1]$. Let K be a continuous function on $[0,1] \times [0,1]$. Define a sequence of functions g_n on $[0,1]$ by

$$g_n(x) := \int_0^1 K(x,y) f_n(y)\, \mathrm{d}y.$$

Show that the sequence g_n contains a uniformly convergent subsequence.

Q.4 Let $\{f_n\}$ be a sequence of measurable functions on a $[0,1]$, and suppose that for every $a > 0$ the infinite series $\sum\limits_{n=1}^{\infty} \mu\left(\{x \in [0,1] \mid |f_n(x)| > a\}\right)$ converges; here μ is the Lebesgue measure. Prove that $\lim f_n(x) = 0$ for almost every $x \in [0,1]$.

Q.5 Let $A \subset \mathbb{R}$ be a set of zero Lebesgue measure. Prove that it can be 'translated completely into the set of irrationals,' that is, there exists a $c \in \mathbb{R}$ such that $A + c \subset \mathbb{R} \setminus \mathbb{Q}$, where $A + c := \{x + c \mid x \in A\}$.

Q.6 Let μ be the Lebesgue measure on the interval $[a,b]$. Let A_n, $n \geq 1$ be measurable subsets of $[a,b]$, and $f(x)$ the number of sets containing x, for $x \in [a,b]$, that is $f(x) = \#(\{n \geq 1 \mid x \in A_n\})$. Prove that $f : [a,b] \to \mathbb{N} \cup \{+\infty\}$ is measurable and that

$$(b-a) \int_{\mathbb{R}} f^2(x)\ \mathrm{d}x \geq \left[\sum_{i=1}^{\infty} \mu(A_i)\right]^2.$$