RETURN THIS COVER SHEET WITH YOUR EXAM AND SOLUTIONS!

Analysis

Ph.D. Preliminary Examination Review

Department of Mathematics

University of Colorado

August 2025

- Answer each of the six questions on a separate page. Turn in a page for each problem even if you cannot solve the problem.
- Label each answer sheet with the problem number.
- **Put your number, not your name, in the upper right hand corner of each page**. If you have not received a number, please choose one (1234 for instance) and notify the Graduate Program Assistant (Kellie Geldreich) as to which number you have chosen.

Problem	1	2	3	4	5	6	Total
Points	17	17	17	17	17	17	102
Score							

Problem 1. Prove or disprove the following statement. There exists a nowhere dense set (that is, a set whose closure has empty interior) in $[0,1]$ which has positive Lebesgue measure.

Problem 2.

Let (X,d) be a compact metric space. Let

$$D = \sup\{d(w,z) : w,z \in X\}.$$

Prove that there exist $x,y \in X$ such that $d(x,y) = D$.

Problem 3. Let f be a Lebesgue-integrable function on $\mathbb{R}^d$. Prove that for every $\varepsilon > 0$, there exists a $\delta > 0$ such that for all measurable A, $m(A) < \delta$ implies

$$\int_A |f(x)| dm(x) < \varepsilon.$$

Problem 4. For $a > 0$, define the shift operator $T_a : L^\infty(\mathbb{R}) \to L^\infty(\mathbb{R})$ as

$$(T_a(f))(x) = f(x+a)$$

Suppose that $g \in L^\infty(\mathbb{R})$ and that g is continuous on $\mathbb{R}$. Is it true that

$$\lim_{a\to 0} \|T_a(g) - g\|_\infty = 0?$$

Prove or give a counter-example (with justification).

Problem 5. Let $f : \mathbb{C}\backslash\{0\} \to \mathbb{C}$ be a continuous function. For $n \in \mathbb{Z}$ define the functions $a_n : (0,\infty) \to \mathbb{C}$ as the Fourier coefficients of the functions f on the circle of radius r i.e.

$$f(re^{i\theta}) = \sum_{n\in\mathbb{Z}} a_n(r) e^{in\theta}$$

for $r \in (0,\infty)$ and $\theta \in [0, 2\pi]$. For each $n \in \mathbb{Z}$, prove that the function a_n is continuous on $(0,\infty)$.

Hint: Recall that for each $n \in \mathbb{Z}$ and $r \in (0, \infty)$ we have the formula

$$a_n(r) = \frac{1}{2\pi} \int_0^{2\pi} f(re^{i\theta}) e^{-in\theta} d\theta$$

Problem 6. Consider the complex Hilbert space $L^2([0, 2\pi])$. Let $V \subset L^2([0, 2\pi])$ be the vector space defined as

$$V = \left\{ \sum_{n=1}^{N} (a_n \cos(nx) + b_n \sin(nx)) \middle| N \in \mathbb{N}, a_n, b_n \in \mathbb{C} \right\}$$

Let $\overline{V}$ be the closure of V in the Hilbert space $L^2([0, 2\pi])$. Prove that $\overline{V} \neq L^2([0, 2\pi])$.