RETURN THIS COVER SHEET WITH YOUR EXAM AND SOLUTIONS!

Analysis

Ph.D. Preliminary Examination

Department of Mathematics

University of Colorado

August, 2024

- Answer each of the six questions on a separate page. Turn in a page for each problem even if you cannot solve the problem.
- Label each answer sheet with the problem number.
- **Put your number, not your name, in the upper right hand corner of each page**. If you have not received a number, please choose one (1234 for instance) and notify the Graduate Program Assistant (Kellie Geldreich) as to which number you have chosen.

Problem	1	2	3	4	5	6	Total
Points	17	17	17	17	17	17	102
Score							

Problem 1. Let S and T be nonempty subsets of $\mathbb{R}$, and suppose $\sigma \leq \tau$ for all $\sigma \in S, \tau \in T$. Prove that $\sup S \leq \inf T$.

Problem 2. If f, g are nonnegative integrable functions on $[0,1]$ such that $fg \geq 1$, then $\int f d\mu \cdot \int g d\mu \geq 1$ where μ stands for Lebesgue measure.

Problem 3. For $N = 1, 2, 3, \ldots$, let $f_N \in L^2(0,1)$ be defined by

$$f_N(x) = \begin{cases} 0 & \text{if } x = 0, \\ \sqrt{N}(1 - Nx) & \text{if } 0 < x \leq \dfrac{1}{N}, \\ 0 & \text{if } \dfrac{1}{N} < x \leq 1. \end{cases}$$

Also let $f(x) = 0$ for all $x \in [0,1]$.

(a) Does f_N converges pointwise to f on $[0,1]$?

(b) Compute $||f_N - f||_2$ (here, $||\cdot||_2$ denotes the $L^2(0,1)$ norm).

(c) Does f_N converge in the norm $||\cdot||_2$ to f?

(d) Does f_N converge uniformly to f on $[0,1]$?

Problem 4. Let ϕ_n be a complete orthonormal system in the space $L^2[0,1]$.

(a) Let $B \subset [0,1]$ be a set of strictly positive Lebesgue measure. Show that

$$1 \leq \int_B \sum_1^\infty |\phi_n|^2 \, d\mu$$

(b) Show that $\sum_1^\infty |\phi_n|^2 = \infty$ a.e.

Problem 5. Let $f, g \in L^2(\mathbb{R})$. We define the *convolution* $f * g(x)$ by the Lebesgue integral

$$f * g(x) = \int_{\mathbb{R}} f(x-y) g(y) \, dy,$$

for any $x \in \mathbb{R}$ such that the integral on the right exists in the Lebesgue sense. (You may assume that the function $y \mapsto f(x-y)g(y)$ is, for any x, measurable as a function of y, whenever f and g are measurable.)

Show that, assuming $f, g \in L^2(\mathbb{R})$:

(a) $f * g(x)$ exists, and

$$|f * g(x)| \leq ||f||_2 \cdot ||g||_2,$$

for all $x \in \mathbb{R}$.

(b) If f and g are in fact continuous and compactly supported, then so is $f * g$.

(c)

$$\lim_{x \to \pm\infty} f * g(x) = 0.$$

Hint: use part (b) above.

Problem 6. Let f be an integrable function on $[0, 1]$ such that $f(x) > 0$ for all x, and let $\alpha > 0$. Show that

$$\lim_{n \to \infty} \int_0^1 n \log(1 + (\frac{f}{n})^\alpha) \, d\mu = \begin{cases} \infty \text{ if } 0 < \alpha < 1 \\ \int_0^1 f \, d\mu \text{ if } \alpha = 1 \\ 0 \text{ if } \alpha > 1 \end{cases}$$

where μ stands for the Lebesgue measure.