RETURN THIS COVER SHEET WITH YOUR EXAM AND SOLUTIONS!

# Analysis

## Ph.D. Preliminary Examination

## Department of Mathematics

## University of Colorado

August, 2023

- Answer each of the six questions on a separate page. Turn in a page for each problem even if you cannot solve the problem.
- Label each answer sheet with the problem number.
- **Put your number, not your name, in the upper right hand corner of each page**. If you have not received a number, please choose one (1234 for instance) and notify the Graduate Program Assistant (Kellie Geldreich) as to which number you have chosen.

| Problem | 1 | 2 | 3 | 4 | 5 | 6 | Total |
|---|---|---|---|---|---|---|---|
| Points | 17 | 17 | 17 | 17 | 17 | 17 | 102 |
| Score | | | | | | | |

**Problem 1.** Let $(X, \mathcal{M}, \mu)$ be a complete measure space with $\mu(X) = 1$. Fix $1 < p < \infty$, and let $\{f_n\}$ be a sequence of elements of $L^p(X, \mu)$ such that there exists $M > 0$ with $\|f_n\|_p \leq M$ for all $n \in \mathbb{N}$. Suppose that the sequence $\{f_n\}$ converges pointwise almost everywhere to a function $f$ defined on $X$. Prove that $f \in L^p(X, \mu)$.

**Problem 2.** Show that if $f : [a, b] \to \mathbb{R}$ is continuous at (Lebesgue) almost every point, then $f$ is Lebesgue measurable. (Hint: the union of an open set and a null-set is always measurable.)

**Problem 3.** Suppose that $f$ and $g$ are positive Lebesgue measurable functions defined on the unit interval $[0, 1]$ with

$$f(x)g(x) \geq 1, \ \forall x \in [0, 1].$$

Prove that

$$\int_{[0,1]} f(x)dm \cdot \int_{[0,1]} g(x)dm \geq 1.$$

**Problem 4.** Let $H$ be a separable Hilbert space and let $S$ be an orthonormal set of vectors (unit vectors where any two of them are orthogonal). Prove that $S$ is either finite or countably infinite.

**Problem 5.** Let $f$ and $g$ be real-valued Lebegue measurable functions on [0,1], not assumed to be integrable over $[0, 1]$. Let $E = \{(x, y) \in [0, 1] \times [0, 1] : \ f(x) = g(y)\}$.

(a) Prove that $E$ is measurable with respect to the Lebesgue product measure $m \times m$ defined on $[0, 1] \times [0, 1]$.
(Hint: consider the function $F(x, y) = f(x) - g(y)$.)

(b) Suppose in addition that $m \times m(E) = 1$. Prove that there is a real constant $c$ such that $f \equiv g \equiv c, \ m$ a.e. on $[0, 1]$.

**Problem 6.** Let us define the function $f : [0, 1] \to \mathbb{R}$ by $f(0) := 0$ and $f(x) := x \sin(1/x), \ x \neq 0$. Show that $f$ is uniformly continuous but not absolutely continuous.