RETURN THIS COVER SHEET WITH YOUR EXAM AND SOLUTIONS!

Analysis

Ph.D. Preliminary Exam

August 12, 2009

INSTRUCTIONS:

1. Put your number, not your name, in the upper right corner of each page. If you have not received a number, please choose one (1234 for instance) and notify the graduate secretary as to which number you have chosen.
2. There are six problems. Each problem is worth the same number of points.
3. Answer each question on a separate page. Label each answer sheet with the problem number.
4. Turn in a page for each problem even if you cannot do the problem.
5. Turn in any scratch paper used.

Real Analysis Prelim **August 2009**

Problem 1. Show that the real-valued function defined by $f(x)=\dfrac{\sin(x)}{x}$ is not Lebesgue integrable on $[1,\infty)$; however, show that the improper Riemann integral of f is convergent.

Problem 2. A map $T: R^n \to R^n$ is said be a Lipschitz function if there exists a constant c such that $\|T(x)-T(y)\| \le c\|x-y\|$ for all vectors x and y in R^n. Prove that Lipschitz maps take Lebesgue measurable sets to Lebesgue measurable sets. Provide an example (for n=1) to show that continuity is not enough to insure this property. Hint: Let f_1 be the Cantor ternary function and let $f(x)=f_1(x)+x$.

Problem 3. Suppose that f is a real-valued C^∞ function on R with $|f'(x)|\le 1$ for all real x. Suppose in addition that $\int_0^\infty |f(x)|dx<\infty$. Prove that $\lim_{x\to\infty} f(x)=0$. Hint: Let $U(\varepsilon)=\{x: \ |f(x)|>\varepsilon\}$. What do you know about $U(\varepsilon)$? How large can f be on this set?

Problem 4. Let f be a real-valued Lebesgue integrable function on [0, ∞). Define $F(x)=\int_0^\infty f(t)\cos(xt)dt$. Show that F is defined on R and is continuous on R. Show that $\lim_{x\to\infty} F(x)=0$.

Problem 5. Given a real-valued measurable function f defined on a measure space (X,M,μ), define the distribution function $\lambda_f(t)=\mu\{x:|f(x)|>t\}$ for all real $t>0$. Suppose that $\lambda_f(t)<\infty$ for all real $t>0$. Let ϕ be a nonnegative Borel function defined on $[0,\infty)$. Show that $\int \phi(|f(x)|)d\mu(x)=-\int_0^\infty \phi(t)d\lambda_f(t)$.

Hint: Explain why the negative measure defined by the decreasing function λ_f as $\nu(a,b]=\lambda_f(b)-\lambda_f(a)=-\mu\{x: a<|f(x)|\le b\}$ has a unique extension to the Borel subsets of R.

Problem 6. A Lebesgue measurable function $f : R^n \to R$ is said to belong to *weak* L^p if $\mu\left\{x \in R^n : |f(x)| > t\right\} \le Ct^{-p}$ for all $t > 0$ and some constant C. Here μ is Lebesgue measure. Let $1 \le p < r < q < \infty$ and assume that f belongs to both *weak* L^p and *weak* L^q. Prove that $f \in L^r$. Hint: This problem is related to Problem 5.