RETURN THIS COVER SHEET WITH YOUR EXAM AND SOLUTIONS!

Algebra

Ph.D. Preliminary Examination

Department of Mathematics

University of Colorado

January, 2026

- Answer each of the six questions on a separate page. Turn in a page for each problem even if you cannot solve the problem.
- Label each answer sheet with the problem number.
- **Put your number, not your name, in the upper right hand corner of each page**. If you have not received a number, please choose one (1234 for instance) and notify the Graduate Program Assistant (Kellie Geldreich) as to which number you have chosen.

Problem	1	2	3	4	5	6	Total
Points	17	17	17	17	17	17	102
Score							

Problem 1. A subgroup H of a group G is a **maximal** subgroup of G if $H \neq G$ and the only subgroups of G containing H are H and G.

(a) Show that if H is a maximal subgroup of G that is normal in G, then the index of H in G is finite and prime.

(b) Describe all the maximal subgroups of the dihedral group of order $2p$, where p is an odd prime. How many are there?

Problem 2. Let G be a subgroup of the symmetric group S_n whose action on $\{1, \dots, n\}$ is transitive (that is, has only one orbit).

(a) Show that for any $x, y \in \{1, \dots, n\}$ the stabilizer of x in G and the stabilizer of y in G are conjugate in G.

(b) Show that if G is abelian, then $|G| = n$.

(c) Give an example of n and a subgroup G of S_n such that G is transitive and abelian but not cyclic.

Problem 3. A commutative ring R with 1 satisfies the descending chain condition (DCC) on ideals if it has no infinite strictly decreasing chain of ideals $I_1 \supsetneq I_2 \supsetneq I_3 \supsetneq \cdots$.

(a) Show that an integral domain R with DCC on ideals is a field.

(Hint: Consider the chain of principal ideals $(a) \supsetneq (a^2) \supsetneq \dots$ for $a \in R$.)

(b) Show that every prime ideal of a ring R with DCC on ideals is a maximal ideal.

Problem 4.

(a) Find all possible rational canonical forms for a matrix over $\mathbb{F}_3$ with characteristic polynomial $x^4 - 1$.

(b) Find all possible rational canonical forms for a matrix over $\mathbb{F}_2$ with characteristic polynomial $x^4 - 1$.

Problem 5. Let K be a field that is an algebraic extension of its prime subfield, F. Show that any field endomorphism $\alpha \colon K \to K$ is an isomorphism.

Problem 6. Let K/F be a Galois extension whose Galois group is isomorphic to S_3. Prove that K is the splitting field over F of some irreducible cubic polynomial $f \in F[x]$.