RETURN THIS COVER SHEET WITH YOUR EXAM AND SOLUTIONS!

Algebra

Ph.D. Preliminary Examination

Department of Mathematics

University of Colorado

January, 2025

- Answer each of the six questions on a separate page. Turn in a page for each problem even if you cannot solve the problem.
- Label each answer sheet with the problem number.
- **Put your number, not your name, in the upper right hand corner of each page**. If you have not received a number, please choose one (1234 for instance) and notify the Graduate Program Assistant (Kellie Geldreich) as to which number you have chosen.

Problem	1	2	3	4	5	6	Total
Points	17	17	17	17	17	17	102
Score							

Problem 1.

(a) Let G be the group $\mathrm{GL}_2(\mathbb{Z}/3\mathbb{Z})$ of invertible 2×2 matrices with entries in $\mathbb{Z}/3\mathbb{Z}$. Show that $|G| = 48$.

(b) Let H be the subset $\mathrm{SL}_2(\mathbb{Z}/3\mathbb{Z})$ of elements of G whose determinant is 1. Show that H is a subgroup of G of index 2.

(c) Compute the number of elements of G of order 3.

Problem 2. Let p be prime number and let G be a group of order p^e for some natural number $e > 1$. Can G have a conjugacy class of size p^{e-1}? (If Yes, give an example. If No, give an argument.)

Problem 3.

(a) Let R be a commutative ring with identity 1, and S a subring of R. Let P be a prime ideal of R. Show $S \cap P$ is a prime ideal of S.

(b) The ring $\mathbb{Z}[x]$ is a famous example of a UFD which is a non PID, because, e.g., $I = (2, x)$ is not a principal ideal. Show that I is a prime ideal of $\mathbb{Z}[x]$.

(c) Let P be a prime ideal of $\mathbb{Z}[x]$. Show that if $\mathbb{Z} \cap P \neq (0)$, then P is generated as an ideal of $\mathbb{Z}[x]$ by at most 2 elements.

Problem 4.

(a) Call an $n \times n$ matrix A over a field $\mathbb{F}$ a *projection matrix* if $A^2 = A$. Show that a projection matrix is diagonalizable over $\mathbb{F}$.

(b) Assume that A and B are $n \times n$ projection matrices over a field $\mathbb{F}$. Show that there is a matrix B' that is similar to B over $\mathbb{F}$ and commutes with A.

Problem 5.

Let p and q be distinct primes. Show that $\sqrt[q]{p} + \sqrt[p]{q}$ is irrational.

Problem 6.

(a) Let p be an odd prime, ζ be a primitive p^{th}-root of unity, and $K = \mathbb{Q}(\zeta)$. Show that there is a unique subfield L of K such that $[K : L] = 2$.

(b) Show that $L = \mathbb{Q}(\zeta + \zeta^{-1})$.