RETURN THIS COVER SHEET WITH YOUR EXAM AND SOLUTIONS!

Algebra

Ph.D. Preliminary Examination

Department of Mathematics

University of Colorado

January, 2024

- Answer each of the six questions on a separate page. Turn in a page for each problem even if you cannot solve the problem.
- Label each answer sheet with the problem number.
- **Put your number, not your name, in the upper right hand corner of each page**. If you have not received a number, please choose one (1234 for instance) and notify the Graduate Program Assistant (Kellie Geldreich) as to which number you have chosen.

Problem	1	2	3	4	5	6	Total
Points	17	17	17	17	17	17	102
Score							

Problem 1. Let $\mathrm{GL}_2(\mathrm{F}_5)$ be the general linear group of invertible 2×2-matrices over the field F_5 of size 5.

(a) What is the size of a Sylow 5-subgroup of $\mathrm{GL}_2(\mathrm{F}_5)$? Determine one Sylow 5-subgroup explicitly by parametrizing its elements. What is its isomorphism type?

(b) How many Sylow 5-subgroups of $\mathrm{GL}_2(\mathrm{F}_5)$ are there?

Problem 2.

(a) Let G be a group. Show that $G \times G$ is isomorphic to $G \rtimes G$, where G acts on itself by conjugation.

(b) Give an example of a group G and a semidirect product $G \rtimes G$ that is not isomorphic to $G \times G$.

Problem 3. Let p be a prime number and let

$$A = \left\{ \frac{a}{b} \in \mathbb{Q} : a, b \in \mathbb{Z} \text{ and } p \text{ does not divide } b \right\}.$$

(a) Find all (multiplicative) units of the subring A of $\mathbb{Q}$.

(b) Use (a) to directly find all prime ideals of A without resorting to any general structure theorems on prime ideals of A.

Problem 4. Let R be a principal ideal domain, let M, N be free R-modules of finite rank, and let $\varphi\colon M \to N$ be an R-module homomorphism.

(a) Show that $\ker \varphi$ is a direct summand in M.

[Hint: Recall that every submodule of a free module M over a PID is free (of rank at most the rank of M).]

(b) Give an example to show that $\varphi(M)$ is not necessarily a direct summand in N.

Problem 5. Let K be a field of characteristic 0 with algebraic closure $\overline{K}$ and let $f(x) \in K[x] \setminus K$. Let $g(x) = \gcd(f(x), f'(x))$ in $K[x]$ and let $F(x) = \frac{f(x)}{g(x)}$. Then show that $f(x)$ and $F(x)$ have the same zeros in $\overline{K}$ and that all the zeros of F in $\overline{K}$ are simple (= of multiplicity 1).

Problem 6.

Let $\omega = e^{\frac{2\pi i}{3}}$ and let $K = \mathbb{Q}(\omega, \sqrt[3]{2})$.

(a) Show that $\mathbb{Q}(\sqrt[3]{2})/\mathbb{Q}$ is not Galois.

(b) Show that $f(x) = x^3 - 2$ is irreducible over $\mathbb{Q}(\omega)$.

(c) Show that $K/\mathbb{Q}(\omega)$ is Galois.

(d) Find the Galois group $\mathrm{Gal}(K/\mathbb{Q}(\omega))$ with an explicit enumeration of all its elements.