

RETURN THIS COVER SHEET WITH YOUR EXAM AND SOLUTIONS!

Algebra

Ph.D. Preliminary Examination

Department of Mathematics

University of Colorado

August 2025

- Answer each of the six questions on a separate page. Turn in a page for each problem even if you cannot solve the problem.
- Label each answer sheet with the problem number.
- **Put your number, not your name, in the upper right hand corner of each page.** If you have not received a number, please choose one (1234 for instance) and notify the Graduate Program Assistant (Kellie Geldreich) as to which number you have chosen.

Problem	1	2	3	4	5	6	Total
Points	17	17	17	17	17	17	102
Score							

Problem 1. Prove that every group of order 24 has a normal subgroup of order 4 or a normal subgroup of order 8 (or both).

Problem 2. Let G be a group, and let $\{N_i : i \in I\}$ be the set of all subgroups of G with the property that G/N_i is abelian. Prove that G is abelian if and only if $\bigcap_{i \in I} N_i$ is the trivial subgroup.

Problem 3. Let R be an integral domain that is not a field. Show that every nonzero ideal contains a proper nonzero subideal.

Problem 4. Let p be an odd prime and let $G = GL_p(\mathbb{Q})$. Prove that any two elements of order p in G are conjugate.

Problem 5. Let $f(x) \in \mathbb{Q}[x]$ be a monic irreducible polynomial that has a root α such that $f(\alpha^2) = 0$. Show that $\mathbb{Q}(\alpha)$ is a splitting field for $f(x)$.

Problem 6. Let K be the splitting field of the polynomial $f(x) \in \mathbb{Q}[x]$, where

$$f(x) = x^5 + x^4 - 4x^3 - 3x^2 + 3x + 1.$$

One of the roots of $f(x)$ in K is $2 \cos(2\pi/11)$.

- (a) Find all the roots of $f(x)$ in K .
- (b) Prove that $f(x)$ is irreducible over $\mathbb{Q}$.
- (c) Prove that $|K : \mathbb{Q}| = 5$.