RETURN THIS COVER SHEET WITH YOUR EXAM AND SOLUTIONS!

Algebra

Ph.D. Preliminary Examination

Department of Mathematics

University of Colorado

August, 2024

- Answer each of the six questions on a separate page. Turn in a page for each problem even if you cannot solve the problem.
- Label each answer sheet with the problem number.
- **Put your number, not your name, in the upper right hand corner of each page**. If you have not received a number, please choose one (1234 for instance) and notify the Graduate Program Assistant (Kellie Geldreich) as to which number you have chosen.

Problem	1	2	3	4	5	6	Total
Points	17	17	17	17	17	17	102
Score							

Problem 1. Let p be a prime number and let G be a finite non-trivial p-group.

(a) Prove that G has non-trivial center.

(b) Prove that if G is a non-Abelian group of order p^3, then its center has order p. (Hint: consider the centralizer of a non-central element.)

Problem 2.

Classify groups of order $57 = 3 \cdot 19$ up to isomorphism, with justification.

Problem 3. Give examples, with proof, of:

(a) An ideal I in the principal ideal domain $\mathbb{Z}[i]$ such that the quotient $\mathbb{Z}[i]/I$ is a product of two fields.

(b) A commutative ring that has exactly one maximal ideal and is not a field.

Problem 4.

Classify up to similarity all 3×3 complex matrices A such that $A^3 = I$, with justification.

Problem 5. Let $\mathbb{F} \subset \mathbb{K}$ be a field extension and let $\alpha \in \mathbb{K}$ be an algebraic element of odd degree over $\mathbb{F}$. Show that $\mathbb{F}(\alpha) = \mathbb{F}(\alpha^2)$.

Problem 6.

Let $f(x) = x^3 - 2$ and let $\mathbb{K} = \mathbb{Q}(\sqrt[3]{2}, \zeta_3)$ be the splitting field of $f(x)$ over $\mathbb{Q}$, where ζ_3 is a primitive 3^{rd} root of unity.

(1) Show that $Aut(\mathbb{K}/\mathbb{Q})$ is isomorphic to the symmetric group S_3.

(2) Describe all of the intermediate fields of this extension (that is, describe all fields $\mathbb{E}$ such that $\mathbb{Q} \subseteq \mathbb{E} \subseteq \mathbb{K}$) .

(3) Explain why your list of intermediate fields is complete.