RETURN THIS COVER SHEET WITH YOUR EXAM AND SOLUTIONS!

Algebra

Ph.D. Preliminary Examination

Department of Mathematics

University of Colorado

August, 2023

- Answer each of the six questions on a separate page. Turn in a page for each problem even if you cannot do the problem.
- Label each answer sheet with the problem number.
- **Put your number, not your name, in the upper right hand corner of each page**. If you have not received a number, please choose one (1234 for instance) and notify the Graduate Program Assistant (Kellie Geldreich) as to which number you have chosen.

Problem	1	2	3	4	5	6	Total
Points	17	17	17	17	17	17	102
Score							

Problem 1. Suppose that G is a group of order 504 $\quad (= 2^3 \cdot 3^2 \cdot 7)$. Show that if G has an element of order 21, then G has a subgroup of index 8. Hint: Explore the possible numbers of Sylow 7-subgroups.

Problem 2. Let G be a finite group and assume that G acts transitively on each of the sets Y and Z. Denote by $G_y = \text{Stab}(y)$ for some $y \in Y$ and $G_z = \text{Stab}(z)$ for some $z \in Z$ some stabilizer subgroups for these actions. Show that the coordinatewise action of G on the product $Y \times Z$ is transitive if and only if $[G : G_y \cap G_z] = [G : G_y][G : G_z]$.

Problem 3. Show that the ideal (x, y) of the polynomial ring $\mathbb{Q}[x, y]$ is not equal to a principal ideal of this ring.

Problem 4. Let V be a finite dimensional vector space over an algebraically closed field. Let $T \colon V \to V$ be a linear transformation, and let $T^* : V^* \to V^*$ be the dual transformation. Show that the Jordan Canonical Form of T is also the Jordan Canonical Form of T^*.

Problem 5. Let $\mathbb{K}$ be a field and let L be its lattice of subfields. Denote the bottom element of L by $\mathbb{P}$ (the prime subfield of $\mathbb{K}$). Show that the following properties of $\mathbb{E} \in L$ are equivalent

(1) The interval $[\mathbb{P}, \mathbb{E}] := \{\mathbb{F} \in L \mid \mathbb{P} \leq \mathbb{F} \leq \mathbb{E}\}$ consisting of the subfields of $\mathbb{E}$ is finite.

(2) L has a finite chain $\mathbb{P} = \mathbb{F}_0 \leq \mathbb{F}_1 \leq \cdots \leq \mathbb{F}_k = \mathbb{E}$ where, for each i, there are no intermediate subfields $\mathbb{F}_i \lneq \mathbb{G} \lneq \mathbb{F}_{i+1}$.

(3) $\mathbb{E}/\mathbb{P}$ is a field extension of finite degree.

Problem 6. Let $\mathbb{K}$ be a (finite) Galois extension of $\mathbb{F}$. Show that

$$\mathcal{A} = \{\alpha \in \mathbb{K} \mid \mathbb{F}[\alpha] \text{ is Galois over } \mathbb{F} \text{ with } \text{Gal}(\mathbb{F}[\alpha]/\mathbb{F}) \text{ abelian}\}$$

is the underlying set of a subfield of $\mathbb{K}$.