

RETURN THIS COVER SHEET WITH YOUR EXAM AND SOLUTIONS!

Algebra

Ph.D. Preliminary Exam

August 2015

INSTRUCTIONS:

1. Answer each question on a separate page. Turn in a page for each problem even if you cannot do the problem.
2. Put your number, not your name, in the upper right hand corner of each page. If you have not received a number, please choose one (1234 for instance) and notify the graduate secretary as to which number you have chosen.
3. There are 6 problems, each worth the same number of points. Please do them all.

1. Show that if the conjugacy classes of a finite group G have size at most 4, then G is solvable.
2. Show that if F is a nontrivial free group, then F has a proper subgroup of finite index.
3. Show that if R is a PID and S is an integral domain containing no subfield, then any homomorphism $\varphi: R \rightarrow S$ is injective.
4. Let A be an $n \times n$ matrix over a field K . Show that if A has exactly one invariant factor, then any matrix B that commutes with A must be a polynomial of A . (That is, show that if $BA = AB$, then $B = p(A)$ for some $p(x) \in K[x]$.)
5. Suppose that $f \in \mathbb{Z}[x]$ is a monic irreducible polynomial of degree 4. Suppose further that there is a complex number α such that both α and α^2 are roots of f . What is f ?
6. Let ζ be a primitive 8th root of unity and let $K = \mathbb{Q}[\zeta]$. Determine $\text{Gal}(K/\mathbb{Q})$ and all the intermediate fields between \mathbb{Q} and K .