RETURN THIS COVER SHEET WITH YOUR EXAM AND SOLUTIONS!

Algebra

Ph.D. Preliminary Exam

August, 2007

INSTRUCTIONS:

1. Answer each question on a separate page. Turn in a page for each problem even if you cannot do the problem.

2. Label each answer sheet with the problem number.

3. Put your number, not your name, in the upper right hand corner of each page. If you have not received a number, please choose one (1234 for instance) and notify the graduate secretary as to which number you have chosen.

Algebra Prelim **August, 2007**

1. Prove that in a group of order 12, any two elements of order 6 must commute.

2. Show that any group of order 105 has an element of order 35.

3. Let R be an integral domain in which every nonzero element factors into a product of finitely many irreducible elements up to a unit. For any $a, b \in R - \{0\}$, define the ideal

$$I_{a,b} := \{x \in R : ax \in (b)\},$$

where (b) is the ideal of R generated by the element b.

Then show that R is a UFD $\Leftrightarrow$ $I_{a,b}$ is principal for any $a, b \in R - \{0\}$.

4. Let R be an associative ring with $1 \neq 0$ and let $N \subseteq M$ be left R-modules.

Suppose that N and M/N are Noetherian. Then show that M is Noetherian.

5. Let $\circ$ be a binary operation on the field $\mathbb{R}$ of real numbers. Show that $\mathbb{R}$ has a countable subfield F with the following properties:

(i) Every positive element of F has a square root.

(ii) Every polynomial of odd degree over F has a root.

(iii) F is closed under $\circ$.

6. Determine the splitting field of the polynomial $x^5 + 2x^4 + 5x^2 + x + 4$ over F_{11} and its Galois group.