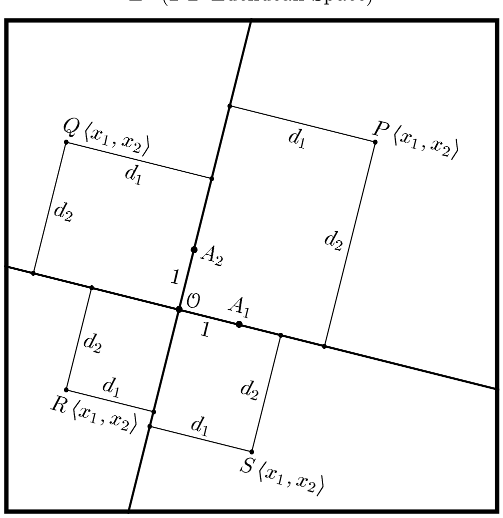
EUCLIDEAN SPACE, CARTESIAN SPACE, ARROWS, AND VECTORS

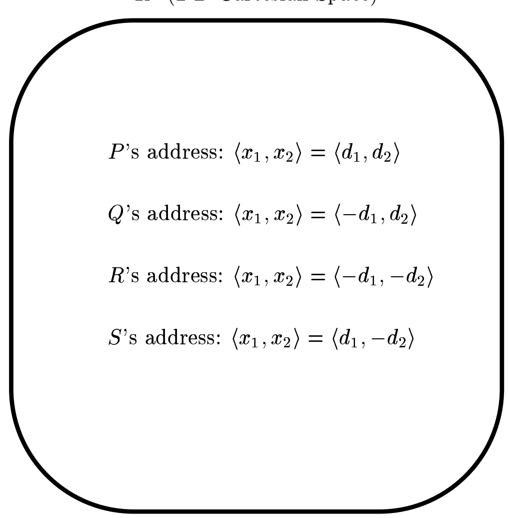
There are two kinds of space used in calculus and analytic geometry: Euclidean Space (ES) and Cartesian Space (CS). Euclidean Space was invented some twenty-three centuries ago, by the Greek mathematician Euclid. Its basic elements are **points**, out of which are constructed lines, planes, circles, triangles, etc. as sets of points having prescribed properties. It comes in dimensionalities 1, 2, and 3 (and higher if you like), denoted by common convention $\mathbb{E}^1$, $\mathbb{E}^2$, and $\mathbb{E}^3$. Cartesian Space was invented four centuries ago by, by the French mathematician René Descartes. It also comes in dimensionalities 1, 2, and 3 (and higher), denoted by $\mathbb{R}^1$, $\mathbb{R}^2$, and $\mathbb{R}^3$. The elements of $\mathbb{R}^1$ are ordered number singles $\langle x_1 \rangle$, the elements of $\mathbb{R}^2$ are ordered number pairs $\langle x_1, x_2 \rangle$, and the elements of $\mathbb{R}^3$ are ordered number triples $\langle x_1, x_2, x_3 \rangle$. In modern terminology these are called **1-vectors**, **2-vectors**, and **3-vectors** (**vectors**, for short).

Descartes' important contribution to the study of Euclidean Space was the construction of **cartesian coordinate systems** (ccsyss, for short), which use very specific procedures to associate with each point of an ES a vector of the CS of the same dimensionality. The procedure for constructing a ccsys for $\mathbb{E}^2$, for example, is this: choose a point $\mathcal{O}$, a point A_1 whose distance from $\mathcal{O}$ is 1, and a point A_2 whose distance from $\mathcal{O}$ is 1 and is such that the angle $A_1\mathcal{O}A_2$ is a right angle. If P is a point of $\mathbb{E}^2$, then associate with P the 2-vector $\langle x_1, x_2 \rangle$ such that x_1 is the distance from P to the line $\overline{\mathcal{O}A_2}$ if P and A_1 are on the same side of $\overline{\mathcal{O}A_2}$, is 0 if P is on $\overline{\mathcal{O}A_2}$, and is the negative of the distance from P to $\overline{\mathcal{O}A_2}$ if P and A_1 are on opposite sides of $\overline{\mathcal{O}A_2}$, x_2 being defined the same way, but with the roles of A_1 and A_2 reversed. The point $\mathcal{O}$ is the **origin** of the ccsys, the line $\overline{\mathcal{O}A_1}$ is the **1-axis** (also, **x-axis**) of the ccsys, and the line $\overline{\mathcal{O}A_2}$ is the **2-axis** (also, **y-axis**) of the ccsys. An analogous but more complex procedure involving points $\mathcal{O}$, A_1, A_2, and A_3 is used for $\mathbb{E}^3$, and a simpler one for $\mathbb{E}^1$. For each ES there are many different cartesian coordinate systems. If P is a point of $\mathbb{E}^2$, then in each ccsys for $\mathbb{E}^2$ the vector $\langle x_1, x_2 \rangle$ in $\mathbb{R}^2$ associated with P is the **address** of P, and x_1 and x_2 are the **coordinates** of P, *in that coordinate system.* The same terminology applies to $\mathbb{E}^1$ and $\mathbb{E}^3$.

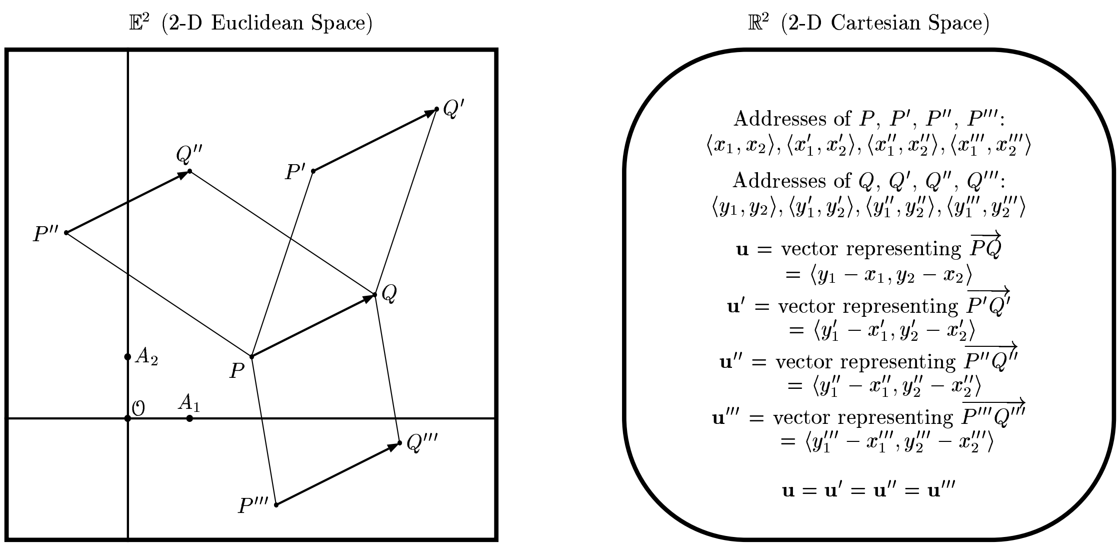
$\mathbb{E}^2$ (2-D Euclidean Space)



$\mathbb{R}^2$ (2-D Cartesian Space)



The invention of cartesian coordinate systems made possible the study of objects in ES such as lines, circles, ellipses, parabolas, etc. by study of algebraic equations satisfied by the coordinates of the points of each such object. The study of these equations, called Analytic Geometry, greatly facilitates the study of the Euclidean Geometry objects, which is what Descartes had in mind. A type of object in ES that it is particularly useful to study in this way is the **arrowvector**, (**arrow**, for short). An arrowvector has an **initial point** (or **base**) and a **terminal point** (or **tip**), and these fully characterize it. The arrowvector with initial point P and terminal point Q is denoted by $\overrightarrow{PQ}$; its **direction** is from P towards Q and its **length**, denoted by $|\overrightarrow{PQ}|$, is $d(P,Q)$ (the distance from P to Q). The arrowvector $\overrightarrow{PQ}$ and the arrowvector $\overrightarrow{P'Q'}$ are **equivalent** if and only if the quadrilateral $PQQ'P'$ is a (possibly degenerate) parallelogram. In a ccsys every arrowvector is represented by a vector, and every vector represents many arrowvectors. If the address of P is $\langle x_1, x_2\rangle$ (in 3-D, $\langle x_1, x_2, x_3\rangle$), and the address of Q is $\langle y_1, y_2\rangle$ (in 3-D, $\langle y_1, y_2, y_3\rangle$), then the vector that represents the arrowvector $\overrightarrow{PQ}$ is $\langle y_1 - x_1, y_2 - x_2\rangle$ (in 3-D, $\langle y_1 - x_1, y_2 - x_2, y_3 - x_3\rangle$). An arrowvector $\overrightarrow{PQ}$ and an arrowvector $\overrightarrow{P'Q'}$ are represented by the the same vector if and only if they are equivalent.



There are two ways to add arrowvectors, corresponding to the two ways they are used to represent physical entities. When the arrowvector $\overrightarrow{PQ}$ represents a displacement (i.e., a move) from the point P to the point Q, and $\overrightarrow{QR}$ represents a further displacement from Q to R, then the **combined displacement** from P to R is represented by the arrow $\overrightarrow{PR}$, which is called the **sum** of $\overrightarrow{PQ}$ and $\overrightarrow{QR}$ and is denoted by $\overrightarrow{PQ} + \overrightarrow{QR}$; thus $\overrightarrow{PQ} + \overrightarrow{QR} := \overrightarrow{PR}$. When the arrowvector $\overrightarrow{PQ}$ represents a velocity, an acceleration, or a force, and $\overrightarrow{PS}$ represents an entity of the same type associated with the same point P, then the **resultant** of these entities is represented by the arrow $\overrightarrow{PR}$ such that the quadrilateral $PQRS$ is a (possibly degenerate) parallelogram. The arrow $\overrightarrow{PR}$ is called the **sum** of $\overrightarrow{PQ}$ and $\overrightarrow{PS}$, and is denoted by $\overrightarrow{PQ} + \overrightarrow{PS}$; thus $\overrightarrow{PQ} + \overrightarrow{PS} := \overrightarrow{PR}$. Note that velocities, accelerations, and, according to Newton, forces are derivatives with respect to time (of, respectively, displacements, velocities, and momenta), but displacements are not derivatives.

If each of $\mathbf{u}$ and $\mathbf{v}$ is a 2-vector (or 3-vector), say $\mathbf{u} = \langle u_1, u_2\rangle$ and $\mathbf{v} = \langle v_1, v_2\rangle$ (or $\mathbf{u} = \langle u_1, u_2, u_3\rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3\rangle$), then the **sum** of $\mathbf{u}$ and $\mathbf{v}$, denoted by $\mathbf{u} + \mathbf{v}$, is the vector $\langle u_1 + v_1, u_2 + v_2\rangle$ (or $\langle u_1 + v_1, u_2 + v_2, u_3 + v_3\rangle$). If the vector $\mathbf{u}$ represents the arrowvector $\overrightarrow{PQ}$, and the vector $\mathbf{v}$ represents the arrowvector $\overrightarrow{QR}$, then the vector $\mathbf{u} + \mathbf{v}$ represents $\overrightarrow{PQ} + \overrightarrow{QR}$. If $\mathbf{v}$ represents the arrowvector $\overrightarrow{PS}$, then $\mathbf{u} + \mathbf{v}$ represents $\overrightarrow{PQ} + \overrightarrow{PS}$.

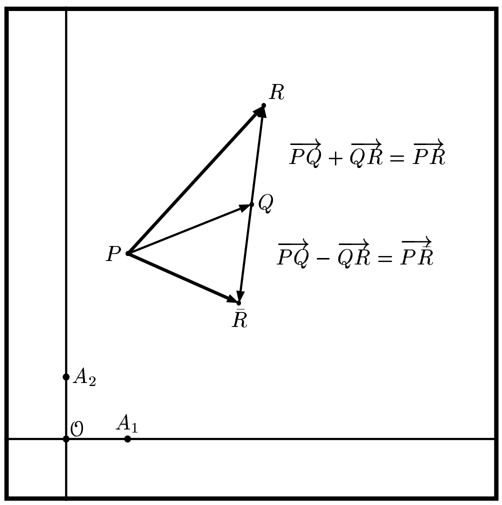
The **negative** of the arrowvector $\overrightarrow{PQ}$, denoted by $-\overrightarrow{PQ}$, is the arrowvector $\overrightarrow{P\bar{Q}}$ such that P is the midpoint of the line interval $Q\bar{Q}$. The **difference** of $\overrightarrow{PQ}$ and $\overrightarrow{QR}$, denoted by $\overrightarrow{PQ} - \overrightarrow{QR}$, is the sum of $\overrightarrow{PQ}$ and $-\overrightarrow{QR}$, i.e., $\overrightarrow{PQ} + -\overrightarrow{QR}$. This says that $\overrightarrow{PQ} - \overrightarrow{QR} = P\bar{R}$, where $\overrightarrow{Q\bar{R}} = -\overrightarrow{QR}$. The

difference of $\overrightarrow{PQ}$ and $\overrightarrow{PS}$, denoted by $\overrightarrow{PQ} - \overrightarrow{PS}$, is the sum of $\overrightarrow{PQ}$ and $-\overrightarrow{PS}$, i.e., $\overrightarrow{PQ} + -\overrightarrow{PS}$. This says that $\overrightarrow{PQ} - \overrightarrow{PS} = P\bar{R}$, where $\overrightarrow{P\bar{R}} = -\overrightarrow{PS}$.

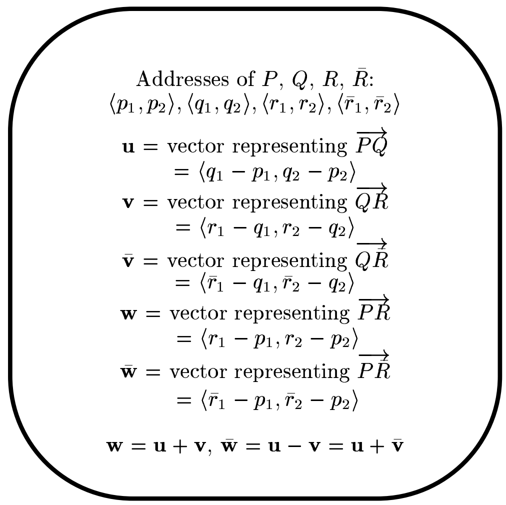
If **u** is a 2-vector (3-vector), say $\mathbf{u} = \langle u_1, u_2 \rangle$ (or $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$), then the **negative** of **u**, denoted by $-\mathbf{u}$, is the vector $\langle -u_1, -u_2 \rangle$ (or $\langle -u_1, -u_2, -u_3 \rangle$). If each of **u** and **v** is a vector, then the **diffence** of **u** and **v**, denoted by $\mathbf{u} - \mathbf{v}$, is the sum of **u** and $-\mathbf{v}$, i.e., $\mathbf{u} + -\mathbf{v}$.

If the vector **u** represents the arrow $\overrightarrow{PQ}$, then the arrow $-\overrightarrow{PQ}$ is represented by $-\mathbf{u}$. If the vector **u** represents the arrow $\overrightarrow{PQ}$, and the vector **v** represents the arrow $\overrightarrow{QR}$, then the vector $\mathbf{u} - \mathbf{v}$ represents $\overrightarrow{PQ} - \overrightarrow{QR}$. If **v** represents $\overrightarrow{PS}$, then $\mathbf{u} - \mathbf{v}$ represents $\overrightarrow{PQ} - \overrightarrow{PS}$.

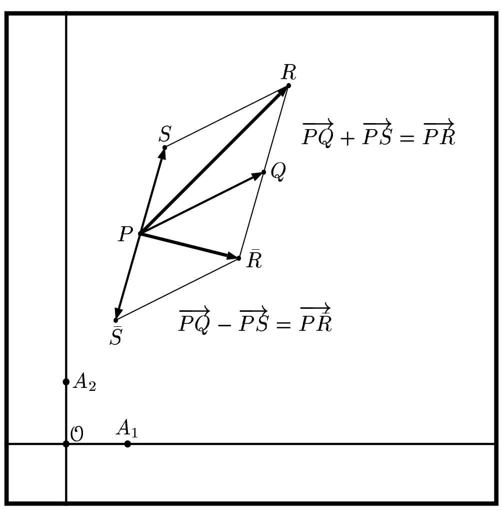
$\mathbb{E}^2$ (2-D Euclidean Space)



$\mathbb{R}^2$ (2-D Cartesian Space)



$\mathbb{E}^2$ (2-D Euclidean Space)



$\mathbb{R}^2$ (2-D Cartesian Space)