Solutions

MATH 2400: CALCULUS 3

10:30 am - 1pm, Wed. May 4, 2016

FINAL EXAM

I have neither given nor received aid on this exam.

Name:______________________________

Check one below !

- ◯ **001** WATTS ....(9AM)
- ◯ **002** GREEN ....(10AM)
- ◯ **003** BLAKESTAD ....(11AM)
- ◯ **004** MISHEV ....(12PM)
- ◯ **005** WASHABAUGH ....(1PM)
- ◯ **006** BULIN ....(2PM)
- ◯ **007** CHHAY ....(3PM)

Notes, electronic devices, and any other aids are **not** permitted on this exam.

If you have a question raise your hand and remain seated. In order to receive full credit your answer must be **complete**, **logical**, **legible**, and **correct**. Show all of your work, and give adequate explanations. No shown work even with the correct final answer will lead to no points. Only give one answer to each problem! If there are two different answers to one problem, the lower score will be chosen.

DO NOT WRITE IN THIS BOX!

Problem	Points	Score
1	12 pts	
2	13 pts	
3	12 pts	
4	13 pts	
5	12 pts	
6	12 pts	
7	13 pts	
8	13 pts	
TOTAL	100 pts	

1. **(12 points)**

The following questions are true/false or multiple-choice. No partial credit will be given and no work is required to be shown on this problem only. Circle your answer.

(a) Let $\vec{v}$ and $\vec{w}$ be two nonzero and nonparallel vectors. Let θ be the angle between $\vec{v}$ and $\vec{w}$ with $0 < \theta < \pi$. Then the area of the parallelogram determined by $\vec{v}$ and $\vec{w}$ is given by

(i) $\vec{v} \times \vec{w}$ (ii) $\|\vec{v}\|\|\vec{w}\| \cos\theta$ (iii) $\|\vec{v}\|\|\vec{w}\| \sin\theta$

(b) Two nonzero vectors $\vec{v}$ and $\vec{w}$ are perpendicular if and only if $\vec{v} \cdot \vec{w} = 0$.

(i) True (ii) False

(c) If f is a continuous function on a closed and bounded set D in $\mathbb{R}^2$, then f attains both an absolute maximum value and an absolute minimum value on D.

(i) True (ii) False

(d) If $\vec{F}$ is a vector field in space whose components have continuous second-order partial derivatives, then $\nabla(\operatorname{curl} \vec{F})$

(i) does not make sense (ii) makes sense and is always zero

(iii) makes sense and may be nonzero

(e) If f is a function of three variables that has continuous second-order partial derivatives, then $\operatorname{curl}(\nabla f)$

(i) does not make sense (ii) makes sense and is always zero

(iii) makes sense and may be nonzero

(f) If f is a function of three variables that has continuous second-order partial derivatives, then $\operatorname{div}(\nabla f)$

(i) does not make sense (ii) makes sense and is always zero

(iii) makes sense and may be nonzero

2. **(13 points)** If the limit exists, evaluate it. If not, explain why not.

(a) **(5 points)**

$$\lim_{(x,y)\to(1,2)} \frac{y^3 - 4x}{x^3 + 4y^3}$$

The point (1,2) is in the domain of the rational function $f(x,y) = \dfrac{y^3-4x}{x^3+4y^3}$. Therefore, f is continuous at (1,2) and we have

$$\lim_{(x,y)\to(1,2)} \frac{y^3-4x}{x^3+4y^3} = \frac{2^3-(4)(1)}{1^3+4(2^3)} = \frac{8-4}{1+32} = \frac{4}{33}$$

(b) **(8 points)**

$$\lim_{(x,y)\to(0,0)} \frac{2x^4 - 3y^4}{x^4 + y^4}$$

As $(x,y)\to(0,0)$ along the x-axis,

$$\frac{2x^4-3y^4}{x^4+y^4} = \frac{2x^4}{x^4} = 2 \longrightarrow 2$$

As $(x,y)\to(0,0)$ along the y-axis,

$$\frac{2x^4-3y^4}{x^4+y^4} = \frac{-3y^4}{y^4} = -3 \longrightarrow -3$$

Since the above two limits are different,

$$\lim_{(x,y)\to(0,0)} \frac{2x^4-3y^4}{x^4+y^4}$$ does not exist.

3. **(12 points)**

Let S be the surface

$$\frac{x^2}{2} + \frac{y^2}{4} + \frac{z^2}{2} = 1.$$

(a) **(7 points)** Find the equation of the tangent plane to the surface S at the point $(1, 0, -1)$.

Let $F(x,y,z) = \frac{x^2}{2} + \frac{y^2}{4} + \frac{z^2}{2}$. Then S is the level surface $F(x,y,z) = 1$.

We have $\vec{\nabla} F = \langle x, \frac{y}{2}, z \rangle$. Then $\vec{\nabla} F(1,0,-1) = \langle 1, 0, -1 \rangle$ is normal to the tangent plane.

The equation of the tangent plane is

$$(1)(x-1) + (0)(y-0) + (-1)(z+1) = 0$$

$$x - z = 2$$

(b) **(5 points)** Find the angle between the tangent plane in part (a) and the plane $y - z = 1$.

A normal vector to the tangent plane $x - z = 2$ in part (a) is $\vec{n_1} = \langle 1, 0, -1 \rangle$. A normal vector to the plane $y - z = 1$ is $\vec{n_2} = \langle 0, 1, -1 \rangle$. The angle between the two planes is

$$\arccos\left(\frac{\vec{n_1} \cdot \vec{n_2}}{\|\vec{n_1}\| \, \|\vec{n_2}\|}\right) = \arccos\left(\frac{1}{\sqrt{2}\,\sqrt{2}}\right)$$

$$= \arccos\left(\frac{1}{2}\right) = \frac{\pi}{3}$$

4. **(13 points)** Find and classify as local maximum, local minimum, or saddle point the critical points of

$$f(x, y) = x^3 - 2xy + \frac{y^2}{2}.$$

$f_x = 3x^2 - 2y = 0$

$f_y = -2x + y = 0$

$f_y = -2x + y = 0 \Rightarrow y = 2x$

Hence $f_x = 3x^2 - 2y = 3x^2 - 2(2x) = 0$

$3x^2 - 4x = 0$

$x(3x - 4) = 0$

$x = 0$ or $x = \frac{4}{3}$

when $x = 0$, $y = 2x = 0$

when $x = \frac{4}{3}$, $y = 2x = \frac{8}{3}$

The critical points are $(0,0)$ and $(\frac{4}{3}, \frac{8}{3})$.

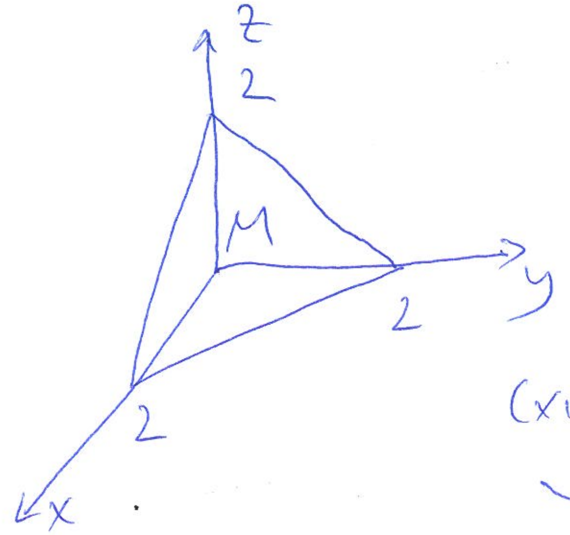
$f_{xx} = 6x$, $f_{yy} = 1$, $f_{xy} = -2$

$D = f_{xx} f_{yy} - f_{xy}^2 = 6x - 4$

$D(0,0) = -4 < 0$ and so we have a saddle point at $(0,0)$

$D(\frac{4}{3}, \frac{8}{3}) = 4 > 0$ and $f_{xx}(\frac{4}{3}, \frac{8}{3}) = 8 > 0$. Therefore, we have a local minimum at $(\frac{4}{3}, \frac{8}{3})$.

5. **(12 points)** Compute $\iint_M (x+y)dS$ where M is the part of the plane $x+y+z=2$ in the first octant.

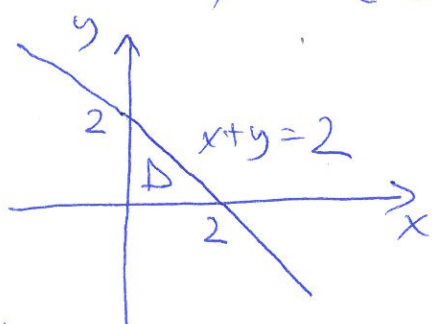


$z = 2-x-y$

We parameterize M by

$\vec{r}(x,y) = \langle x, y, 2-x-y \rangle$,

$(x,y) \in D$, where D is the region below:



$$\vec{r}_x \times \vec{r}_y = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{vmatrix} = \langle 1, 1, 1 \rangle$$

$$\|\vec{r}_x \times \vec{r}_y\| = \sqrt{3}$$

$$\iint_M (x+y)\,dS = \iint_D (x+y)\sqrt{3}\,dy\,dx$$

$$= \sqrt{3}\int_0^2 \int_0^{2-x} (x+y)\,dy\,dx = \sqrt{3}\int_0^2 \left(\left(xy + \frac{y^2}{2}\right)\Big|_{y=0}^{2-x} \right) dx$$

$$= \sqrt{3}\int_0^2 \left(x(2-x) + \frac{(2-x)^2}{2} \right) dx = \sqrt{3}\int_0^2 \left(2 - \frac{x^2}{2}\right) dx$$

$$= \sqrt{3}\left(\left(2x - \frac{x^3}{6}\right)\Big|_0^2 \right) = \sqrt{3}\left(4 - \frac{4}{3}\right) = \frac{8\sqrt{3}}{3}$$

6. **(12 points)** Consider the vector field $\mathbf{F}(x,y) = \langle y^2, 2xy + x\rangle$.

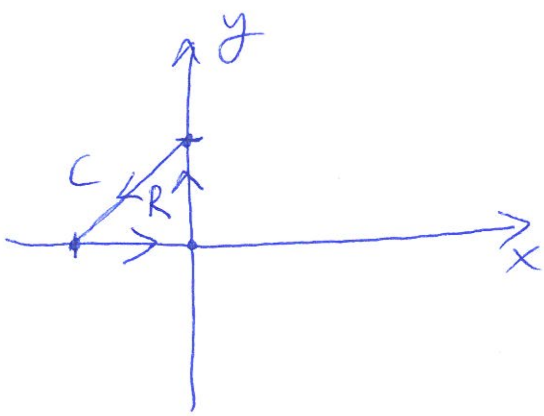
(a) **(5 points)** Is $\mathbf{F}$ conservative? If yes, find a potential function f for it. If not, justify your answer.

$$\frac{\partial}{\partial x}(2xy + x) = 2y + 1 \neq 2y = \frac{\partial}{\partial y}(y^2)$$

Thus, $\vec{F}$ is not conservative

(b) **(7 points)** Let C be the positively oriented boundary of the triangle with vertices $(0,0)$, $(0,1)$ and $(-1,0)$. Evaluate the line integral

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C y^2\,dx + (2xy + x)\,dy.$$



By Green's Theorem,

$$\int_C \vec{F} \cdot d\vec{r} = \int_C y^2 dx + (2xy + x) dy$$

$$= \iint_R \left(\frac{\partial}{\partial x}(2xy + x) - \frac{\partial}{\partial y}(y^2)\right) dA$$

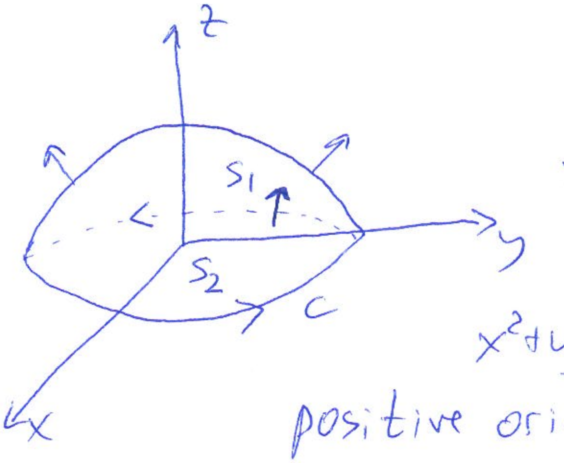
$$= \iint_R (2y + 1 - 2y)\, dA = \iint_R dA$$

$$= \text{Area}(R) = \frac{(1)(1)}{2} = \frac{1}{2}$$

7. (13 points) Let $\mathbf{F}(x, y, z)$ be the vector field

$$\langle xz - y^3\cos(z),\ x^3e^z,\ ze^{x^2+y^2+z^2}\rangle.$$

Find the flux of the **curl** of $\mathbf{F}(x, y, z)$ across the upper hemisphere of $x^2+y^2+z^2 = 1$, oriented upwards. (Use Stokes' Theorem to replace the surface with an easier surface.)



Let S_1 be the upper hemisphere of $x^2+y^2+z^2=1$, oriented upwards. The boundary C is S_1 is the circle $x^2+y^2=1$ in the xy-plane. Then the induced positive orientation of C from S_1 is counterclockwise when viewed from above.

Let S_2 be the disk $x^2+y^2\le 1$, $z=0$ parameterized by $\vec{r}(a,\theta)=\langle a\cos\theta, a\sin\theta, 0\rangle$, $0\le a\le 1$, $0\le\theta\le 2\pi$, We orient S_2 upwards, so that the induced positive orientation of C from S_2 is counterclockwise when viewed from above. We note that

$$\vec{r}_a\times\vec{r}_\theta = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos\theta & \sin\theta & 0 \\ -a\sin\theta & a\cos\theta & 0 \end{vmatrix} = \langle 0,0,a\rangle$$

We need to compute $\iint_{S_1} \text{curl}\,\vec{F}\cdot d\vec{S}$. Applying Stokes' Theorem twice gives $\iint_{S_1}\text{curl}\,\vec{F}\cdot d\vec{S} = \int_C \vec{F}\cdot d\vec{r} = \iint_{S_2}\text{curl}\,\vec{F}\cdot d\vec{S}$

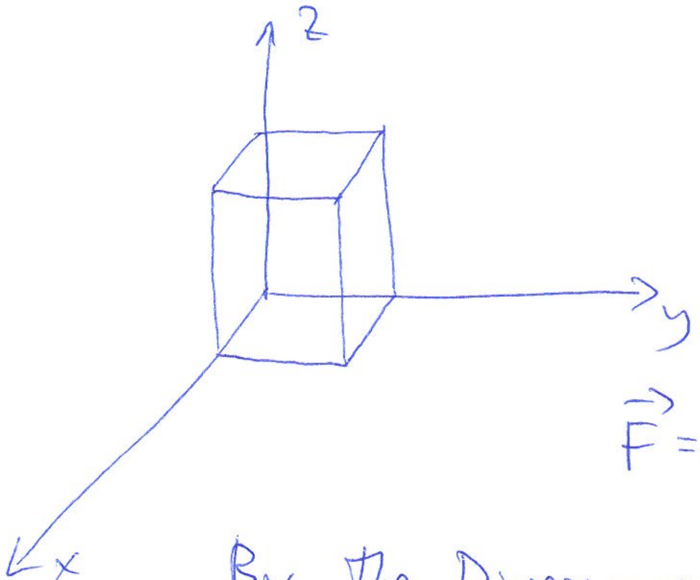
$$= \int_0^{2\pi}\int_0^1 \text{curl}\,\vec{F}\cdot(\vec{r}_a\times\vec{r}_\theta)\,da\,d\theta = \int_0^{2\pi}\int_0^1 \left(3(a\cos\theta)^2e^0 + 3(a\sin\theta)^2\cos(0)\right)a\,da\,d\theta$$

$$= \int_0^{2\pi}\int_0^1 3a^3\,da\,d\theta = \int_0^{2\pi}\left(\frac{3}{4}a^4\Big|_{a=0}^{1}\right)d\theta = \int_0^{2\pi}\frac{3}{4}\,d\theta$$

$$= \left(\frac{3}{4}\right)(2\pi) = \frac{3\pi}{2}$$

8. **(13 points)** Let S be the box with faces $x=0$, $y=0$, $z=0$, $x=2$, $y=3$, and $z=5$, with each face oriented outwards. Compute the integral

$$\iint_S \langle x^4y^2z,\ x^3y^3z,\ x^3y^2z^2\rangle \cdot d\mathbf{S}.$$



Let W be the solid region enclosed by S and let

$$\vec{F} = \langle x^4y^2z, x^3y^3z, x^3y^2z^2\rangle.$$

By the Divergence Theorem,

$$\iint_S \langle x^4y^2z, x^3y^3z, x^3y^2z^2\rangle \cdot d\vec{S} = \iint_S \vec{F}\cdot d\vec{S}$$

$$= \iiint_W \operatorname{div}\vec{F}\, dV = \iiint_W \left(4x^3y^2z + 3x^3y^2z + 2x^3y^2z\right) dV$$

$$= \int_0^2\int_0^3\int_0^5 \left(9x^3y^2z\right) dz\,dy\,dx$$

$$= \int_0^2\int_0^3 \left(\frac{9}{2}x^3y^2z^2\Big|_{z=0}^{5}\right) dy\,dx = \int_0^2\int_0^3 \frac{225}{2}x^3y^2\,dydx$$

$$= \int_0^2 \left(\frac{75}{2}x^3y^3\Big|_{y=0}^{3}\right) dx = \int_0^2 \frac{2025}{2}x^3\,dx$$

$$= \frac{2025}{8}x^4\Big|_0^2 = 4050$$