Solutions

MATH 2400: CALCULUS 3

5:15 - 6:45 pm, Mon. Apr. 11, 2016

MIDTERM 3

I have neither given nor received aid on this exam.

Name:__________________________________

Check one below !

○ **001** WATTS ....(9AM)
○ **002** GREEN .... (10AM)
○ **003** BLAKESTAD .... (11AM)
○ **004** MISHEV .... (12PM)
○ **005** WASHABAUGH .... (1PM)
○ **006** BULIN .... (2PM)
○ **007** CHHAY ....(3PM)

Notes, electronic devices, and any other aids are **not** permitted on this exam.

If you have a question raise your hand and remain seated. In order to receive full credit your answer must be **complete**, **logical**, **legible**, and **correct**. Show all of your work, and give adequate explanations. No shown work even with the correct final answer will lead to no points. Only give one answer to each problem! If there are two different answers to one problem, the lower score will be chosen.

DO NOT WRITE IN THIS BOX!

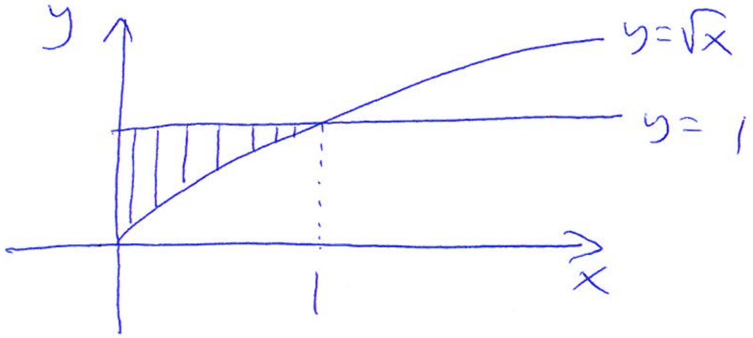
Problem	Points	Score
1	16 pts	
2	17 pts	
3	17 pts	
4	17 pts	
5	16 pts	
6	17 pts	
TOTAL	100 pts	

1. (16 points) Evaluate the integral

$$\iiint\limits_R y\,dV,$$

where R is the finite region bounded by $z = 0$, $x = 0$, $z = 1 - y$ and $y = \sqrt{x}$.

The bounds for z are 0 to $1-y$. The bounds for x and y come from the following shaded region:



We have $\iiint_R y\,dV = \int_0^1 \int_{\sqrt{x}}^1 \int_0^{1-y} y\,dz\,dy\,dx$

$$= \int_0^1 \int_{\sqrt{x}}^1 \left(yz \Big|_{z=0}^{1-y} \right) dy\,dx = \int_0^1 \int_{\sqrt{x}}^1 y(1-y)\,dy\,dx$$

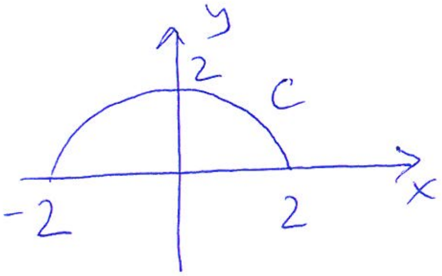
$$= \int_0^1 \int_{\sqrt{x}}^1 (y-y^2)\,dy\,dx = \int_0^1 \left(\left(\frac{y^2}{2} - \frac{y^3}{3} \right) \Big|_{y=\sqrt{x}}^1 \right) dx$$

$$= \int_0^1 \left(\left(\frac{1}{2} - \frac{1}{3} \right) - \left(\frac{x}{2} - \frac{x^{3/2}}{3} \right) \right) dx = \int_0^1 \left(\frac{1}{6} - \frac{x}{2} + \frac{x^{3/2}}{3} \right) dx$$

$$= \left(\frac{1}{6}x - \frac{x^2}{4} + \frac{2}{15}x^{5/2} \right) \Big|_0^1 = \frac{1}{6} - \frac{1}{4} + \frac{2}{15}$$

$$= \frac{10-15+8}{60} = \frac{3}{60} = \frac{1}{20}$$

2. **(17 points)** Suppose that a thin wire is bent into the shape of the semicircle $x^2 + y^2 = 4$, $y \geq 0$. Suppose moreover that the density of the wire is given by $\rho(x, y) = 3 + x$. Find the mass and center of mass of the wire.



We parameterize C by

$x = 2\cos t,\ y = 2\sin t,\ 0 \leq t \leq \pi$

We have $\sqrt{(x'(t))^2 + (y'(t))^2} = \sqrt{(-2\sin t)^2 + (2\cos t)^2}$

$= \sqrt{4\sin^2 t + 4\cos^2 t} = \sqrt{4} = 2$

The mass is given by $m = \int_C (3+x)\,ds = \int_0^{\pi} (3 + 2\cos t)(2)\,dt$

$= 2\left(3t + 2\sin t\right)\Big|_0^{\pi} = 6\pi$

We also have

$\int_C x(3+x)\,ds = \int_0^{\pi} 2\cos t(3 + 2\cos t)(2)\,dt = 4\int_0^{\pi} (3\cos t + 2\cos^2 t)\,dt$

$= 4\int_0^{\pi} \left(3\cos t + 2\left(\frac{1}{2} + \frac{\cos(2t)}{2}\right)\right)dt = 4\int_0^{\pi} \left(1 + 3\cos t + \cos(2t)\right)dt$

$= 4\left(t + 3\sin t + \frac{\sin(2t)}{2}\right)\Big|_0^{\pi} = 4\pi$

and $\int_C y(3+x)\,ds = \int_0^{\pi} 2\sin t(3 + 2\cos t)(2)\,dt = 4\int_0^{\pi} (3\sin t + 2\sin t\cos t)\,dt$

$= 4\left(-3\cos t + \sin^2 t\right)\Big|_0^{\pi} = 4\left(3 - (-3)\right) = 24$

The center of mass is at the point $(\bar{x}, \bar{y})$, where

$\bar{x} = \dfrac{\int_C x(3+x)\,ds}{m} = \dfrac{4\pi}{6\pi} = \dfrac{2}{3}$ and $\bar{y} = \dfrac{\int_C y(3+x)\,ds}{m} = \dfrac{24}{6\pi} = \dfrac{4}{\pi}$

3. **(17 points)** Compute the surface area of the part of the cylinder $x^2+y^2=1$ that lies between the planes $z=0$ and $x+y+z=10$.

We parameterize the surface by

$x=\cos t,\ y=\sin t,\ z=z\ ;\ 0\le t\le 2\pi,\ 0\le z\le 10-\cos t-\sin t$

We have $\vec{r}(t,z)=\langle \cos t, \sin t, z\rangle$,

$$\vec{r}_t\times\vec{r}_z=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ -\sin t & \cos t & 0\\ 0 & 0 & 1\end{vmatrix}=\langle \cos t, \sin t, 0\rangle,$$

$$\|\vec{r}_t\times\vec{r}_z\|=\sqrt{\cos^2 t+\sin^2 t}=1.$$

From here,

$$\text{Surface area}=\int_0^{2\pi}\int_0^{10-\cos t-\sin t}\|\vec{r}_t\times\vec{r}_z\|\,dz\,dt$$

$$=\int_0^{2\pi}\int_0^{10-\cos t-\sin t}dz\,dt=\int_0^{2\pi}(10-\cos t-\sin t)\,dt$$

$$=\left(10t-\sin t+\cos t\right)\Big|_0^{2\pi}$$

$$=(20\pi-0+1)-(0-0+1)=20\pi$$

4. (17 points) Let $\mathbf{F}(x, y) = \langle 3y^2 + 4xe^y,\ 6xy + 2x^2e^y \rangle$.

(a) **(7 points)** Is $\mathbf{F}$ conservative? If yes, find a potential function f for it. If not, justify your answer.

The domain of $\vec{F}$ is all of $\mathbb{R}^2$. Furthermore,

$$\frac{\partial}{\partial x}(6xy + 2x^2e^y) = 6y + 4xe^y = \frac{\partial}{\partial y}(3y^2 + 4xe^y).$$

Hence, $\vec{F}$ is conservative. Let f be a potential function for $\vec{F}$. We have $f_x = 3y^2 + 4xe^y$ and $f_y = 6xy + 2x^2e^y$.

Since $f_x = 3y^2 + 4xe^y$, $f = \int (3y^2 + 4xe^y)\,dx = 3xy^2 + 2x^2e^y + m(y)$

Now, $f_y = \cancel{6xy} + \cancel{2x^2e^y} + m'(y) = \cancel{6xy} + \cancel{2x^2e^y}$. From here,

$m'(y) = 0$, $m(y) = C$. We can take the constant C to be 0. A potential function for $\vec{F}$ is

$$f(x,y) = 3xy^2 + 2x^2e^y.$$

(b) **(10 points)** Let C be the curve starting at $(0, 0)$, following the ellipse $(x-1)^2 + \frac{1}{4}y^2 = 1$ counterclockwise and ending at $(2, 0)$. Compute the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$.

By The Fundamental Theorem for Line Integrals,

$$\int_C \vec{F} \cdot d\vec{r} = \int_C \vec{\nabla} f \cdot d\vec{r} = f(2,0) - f(0,0)$$

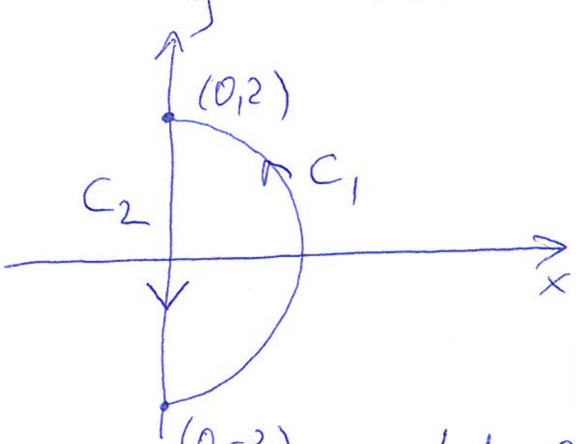
$$= 8 - 0 = 8$$

5. **(16 points)** Let C be the closed curve in the xy-plane consisting of the right half of the circle of radius 2 centered at the origin and the line segment from $(0,2)$ to $(0,-2)$. Suppose C is positively oriented (i.e. C is traversed counterclockwise).

(a) **(8 points)** Evaluate the line integral

$$\int_C y\,dx + (y-x)\,dy$$

directly by parameterizing the curve C.



Let C_1 be the right half of the circle of radius 2 centered at the origin and let C_2 be the line segment from $(0,2)$ to $(0,-2)$. Let C_1 and C_2 have orientations as shown in the picture.

We parameterize C_1 by $x=2\cos t$, $y=2\sin t$, $-\frac{\pi}{2} \le t \le \frac{\pi}{2}$.

We parameterize C_2 by $x=0$, $y=-t$, $-2 \le t \le 2$.

We have

$$\int_C y\,dx+(y-x)\,dy = \int_{C_1} y\,dx+(y-x)\,dy + \int_{C_2} y\,dx+(y-x)\,dy$$

$$= \int_{-\pi/2}^{\pi/2}\Big((2\sin t)(-2\sin t)+(2\sin t-2\cos t)(2\cos t)\Big)dt$$

$$+\int_{-2}^{2}\Big((-t)(0)+(-t-0)(-1)\Big)dt = \int_{-\pi/2}^{\pi/2}\left(-4\sin^2 t-4\cos^2 t+4\sin t\cos t\right)dt$$

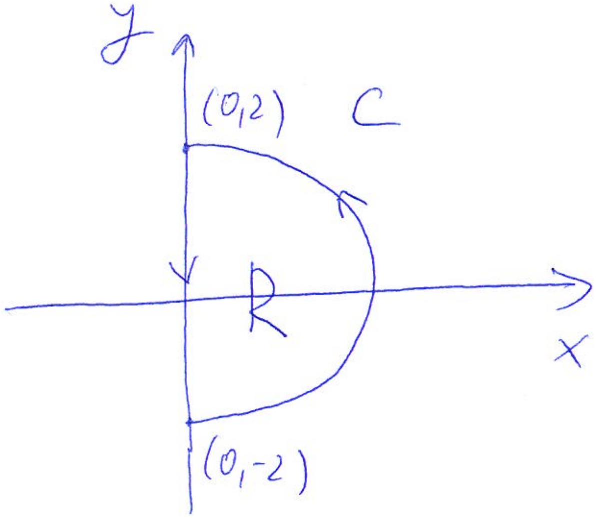
$$+\int_{-2}^{2} t\,dt = \int_{-\pi/2}^{\pi/2}(-4+4\sin t\cos t)\,dt + \int_{-2}^{2} t\,dt$$

$$=\left(-4t+2\sin^2 t\right)\Big|_{-\pi/2}^{\pi/2} + \left(\frac{t^2}{2}\right)\Big|_{-2}^{2} = (-2\pi+2)-(2\pi+2)+2-2 = -4\pi$$

(b) **(8 points)** Evaluate the same line integral

$$\int_C y\,dx + (y - x)\,dy$$

by using Green's Theorem.



Let R be the region enclosed by C. By Green's Theorem,

$$\int_C y\,dx + (y-x)\,dy$$

$$= \iint_R \left(\frac{\partial}{\partial x}(y-x) - \frac{\partial}{\partial y}(y) \right) dA$$

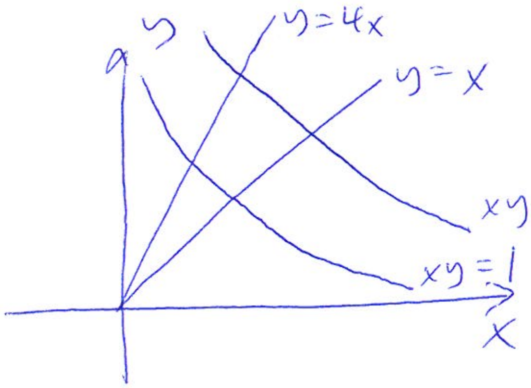
$$= \iint_R (-1-1)\,dA = -2 \iint_R dA$$

$$= -2\,(\text{area}(R)) = (-2)\left(\frac{\pi (2^2)}{2} \right) = -4\pi$$

6. (17 points) Evaluate

$$\iint_R \sqrt{xy^3}\, dA,$$

where R is the region in the xy-plane bounded by $xy = 1$, $xy = 9$, $y = x$ and $y = 4x$.
Hint: Use the substitution $u = xy$, $v = \dfrac{y}{x}$.



Let $u = xy$, $v = \dfrac{y}{x}$. Then $1 \le u \le 9$ and $1 \le v \le 4$.

$$u = xy,\ v = \frac{y}{x} \Rightarrow uv = y^2 \Rightarrow y = \sqrt{uv} = u^{1/2}v^{1/2}$$

$$x = \frac{u}{y} = \frac{u}{\sqrt{uv}} = \sqrt{\frac{u}{v}} = u^{1/2}v^{-1/2}$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \begin{vmatrix} \frac{1}{2}u^{-1/2}v^{-1/2} & -\frac{1}{2}u^{1/2}v^{-3/2} \\ \frac{1}{2}u^{-1/2}v^{1/2} & \frac{1}{2}u^{1/2}v^{-1/2} \end{vmatrix}$$

$$= \frac{1}{4}v^{-1} + \frac{1}{4}v^{-1} = \frac{1}{2v}$$

Thus, $\displaystyle\iint_R \sqrt{xy^3}\, dA = \int_1^4\int_1^9 \sqrt{u^{1/2}v^{-1/2}u^{3/2}v^{3/2}}\left|\frac{1}{2v}\right| du\,dv$

$$= \int_1^4\int_1^9 \sqrt{u^2 v}\left(\frac{1}{2v}\right) du\,dv = \int_1^4\int_1^9 \frac{u}{2\sqrt{v}}\, du\,dv$$

$$= \int_1^4 \left(\frac{u^2}{4\sqrt{v}}\Bigg|_{u=1}^{9}\right) dv = \int_1^4 \frac{20}{\sqrt{v}}\, dv = 40\sqrt{v}\Big|_1^4$$

$$= 40$$