Solutions

MATH 2400: CALCULUS 3

5:15 - 6:45 pm, Mon. Feb. 8, 2016

MIDTERM 1

I have neither given nor received aid on this exam.

Name:________________________________

Check one below !

- ◯ **001** WATTS ....(9AM)
- ◯ **002** GREEN ....(10AM)
- ◯ **003** BLAKESTAD ....(11AM)
- ◯ **004** MISHEV ....(12PM)
- ◯ **005** WASHABAUGH ....(1PM)
- ◯ **006** BULIN ....(2PM)
- ◯ **007** CHHAY ....(3PM)

Notes, electronic devices, and any other aids are **not** permitted on this exam.

If you have a question raise your hand and remain seated. In order to receive full credit your answer must be **complete**, **logical**, **legible**, and **correct**. Show all of your work, and give adequate explanations. No shown work even with the correct final answer will lead to no points. Only give one answer to each problem! If there are two different answers to one problem, the lower score will be chosen.

DO NOT WRITE IN THIS BOX!

Problem	Points	Score
1	17 pts	
2	17 pts	
3	16 pts	
4	17 pts	
5	16 pts	
6	17 pts	
TOTAL	100 pts	

1. (17 points)

(a) **(12 points)** Find an equation of the plane that passes through the three points $P = (1,1,0)$, $Q = (0,2,1)$, and $R = (3,2,-1)$.

$\overrightarrow{PQ} = \langle -1, 1, 1 \rangle$, $\overrightarrow{PR} = \langle 2, 1, -1 \rangle$

$\vec{n} = \overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & 1 & 1 \\ 2 & 1 & -1 \end{vmatrix} = \langle -2, 1, -3 \rangle$ is

normal to the plane.

Using $\vec{n} = \langle -2, 1, -3 \rangle$ and $P = (1,1,0)$, the equation of the plane is

$$-2(x-1) + 1(y-1) - 3(z-0) = 0$$

$$-2x + 2 + y - 1 - 3z = 0$$

$$2x - y + 3z - 1 = 0$$

(b) **(5 points)** Give the parametric equations of the line perpendicular to the plane from part (a) that passes through P.

The vector $\vec{n} = \langle -2, 1, -3 \rangle$ from part (a) is parallel to the line. Thus, parametric equations for the line are

$$x = 1 - 2t$$

$$y = 1 + t$$

$$z = -3t$$

2. **(17 points)**

(a) **(12 points)** Consider the surface S given by the equation

$$x^2 - 2y^2 + z^2 = 1.$$

Sktech the intersection of S with the planes

i. $x = 0$

ii. $y = 0$

iii. $z = 0$

iv. $z = 2$

v. $x = 1$

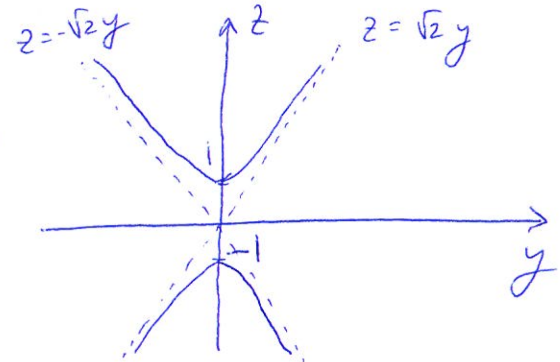
vi. $x = 2$

(If the intersection is a hyperbola, the asymptotes are drawn by a dashed line and are also labeled.)

Remember to label your axes!

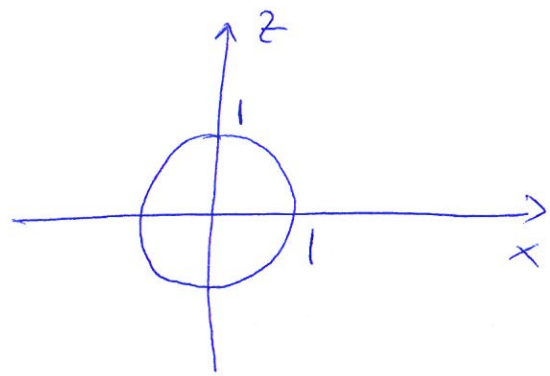
(i)

$x=0 \Rightarrow -2y^2+z^2=1$, This is a hyperbola,



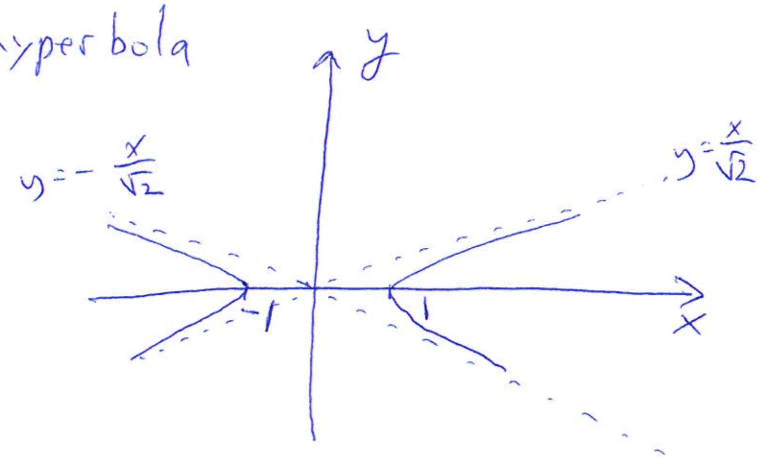
(ii)

$y=0 \Rightarrow x^2+z^2=1$, This is a circle



(iii)

$z=0 \Rightarrow x^2-2y^2=1$, This is a hyperbola

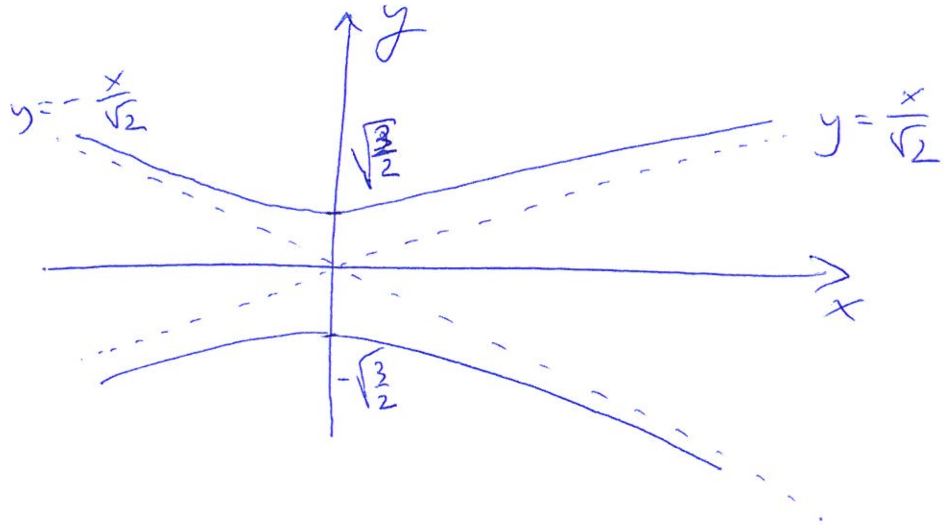


(over $\longrightarrow$)

(iv)

$$z=2 \Rightarrow x^2-2y^2=-3 \Rightarrow -\frac{x^2}{3}+\frac{2}{3}y^2=1$$

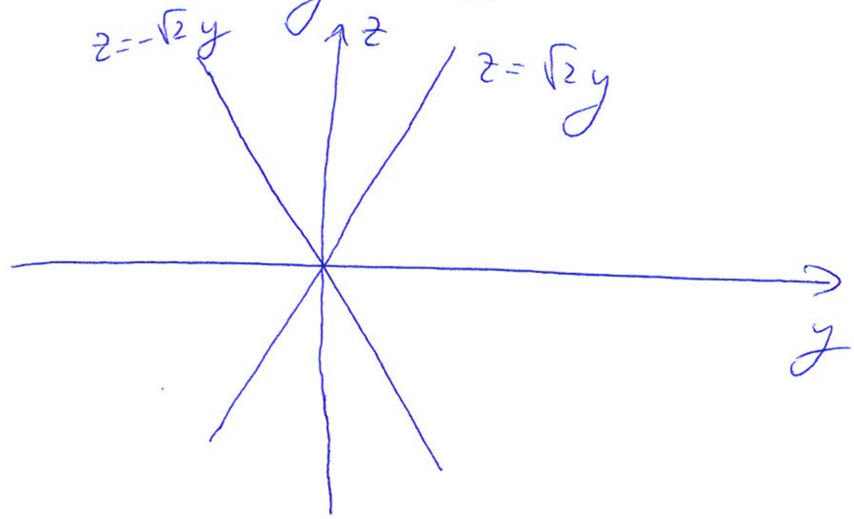
This is a hyperbola



(v)

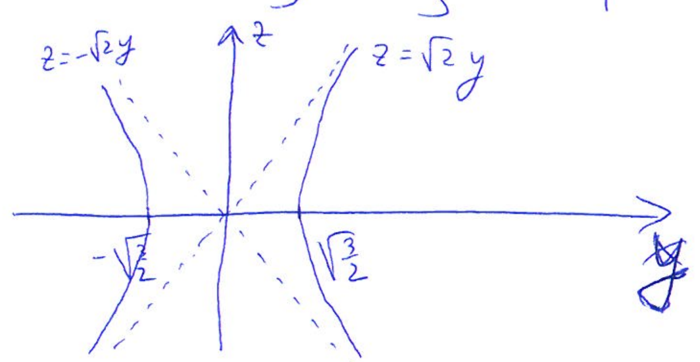
$$x=1 \Rightarrow -2y^2+z^2=0 \Rightarrow z^2=2y^2 \Rightarrow z=\pm\sqrt{2}\,y$$

This is a pair of intersecting lines



(vi) $x=2 \Rightarrow -2y^2+z^2=-3 \Rightarrow \frac{2}{3}y^2-\frac{1}{3}z^2=1$

This is a hyperbola



(b) **5 points** Write down the equation of the paraboloid with apex at $(0,0,0)$ opening in the positive x-direction which intersects the plane $x = 4$ in a circle of radius 3.

The equation will have the form

$$x = ay^2 + bz^2 \quad \text{for some } a > 0, b > 0$$

Since the intersection with the plane $x = 4$ is a circle, we must have $b = a$.

So, now the equation is

$$x = ay^2 + az^2$$

$$\frac{x}{a} = y^2 + z^2$$

Since the intersection with the plane $x = 4$ is a circle of radius 3, we must have

$$\frac{4}{a} = 3^2$$

$$a = \frac{4}{9}$$

Therefore, the equation of the paraboloid is

$$x = \frac{4}{9}(y^2 + z^2)$$

3. (16 points) Let ρ be the plane given by the equation $x + 2y + 3z = 6$ and let L be the line passing through the points $P(1, 0, 0)$ and$Q(-1, 3, 1)$.

(a) **(8 points)** What is the intersection of L and ρ?

$\vec{PQ} = \langle -2, 3, 1 \rangle$ is parallel to L. Using the point $P(1,0,0)$, parametric equations for L are

$x = 1-2t,\ y = 3t,\ z = t$

Now $x+2y+3z=6$ leads to

$$1-2t+2(3t)+3t=6$$
$$1-2t+6t+3t=6$$
$$7t=5$$
$$t=5/7$$

From here, $x\left(\frac{5}{7}\right) = -\frac{3}{7},\ y\left(\frac{5}{7}\right) = \frac{15}{7},\ z\left(\frac{5}{7}\right) = \frac{5}{7}$. The intersection of L and ρ is the point $\left(-\frac{3}{7}, \frac{15}{7}, \frac{5}{7}\right)$

(b) **(8 points)** Compute the angle between L and ρ.

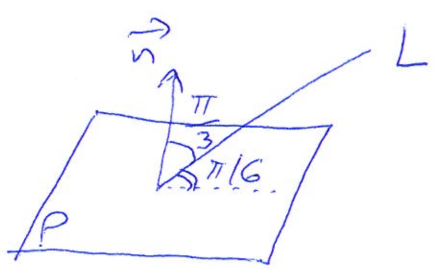
$\vec{PQ} = \langle -2, 3, 1 \rangle$ is parallel to L

$\vec{n} = \langle 1, 2, 3 \rangle$ is perpendicular to ρ

The angle between $\vec{PQ}$ and $\vec{n}$ is $\arccos\left(\dfrac{\vec{PQ}\cdot\vec{n}}{\|\vec{PQ}\|\,\|\vec{n}\|}\right)$

$$= \arccos\left(\frac{7}{\sqrt{14}\sqrt{14}}\right) = \arccos\left(\frac{1}{2}\right) = \frac{\pi}{3}$$

Hence, the angle between L and ρ is $\frac{\pi}{2} - \frac{\pi}{3} = \frac{\pi}{6}$.



4. (17 points)

(a) **(9 points)** Two particles travel along the space curves $\vec{r}_1 = \langle 2t, -t^3, 2t^2 \rangle$ and $\vec{r}_2 = \langle 1+t, -t^2, 1-t \rangle$. Do the particles collide? If so, find the coordinates of the point of collision. If not, explain why they do not collide.

If the particles collide, there is t such that

$$\begin{cases} 2t = 1+t \\ -t^3 = -t^2 \\ 2t^2 = 1-t \end{cases}$$

$2t = 1+t$ gives $t = 1$

$t=1$ satisfies $-t^3 = -t^2$

However, $t=1$ does not satisfy $2t^2 = 1-t$. Therefore, the particles do not collide.

(b) **(8 points)** Find parametric equations for the tangent line to the curve with parametric equations

$$x = t + \sin t, \quad y = t - \cos t, \quad z = te^{-t}$$

at the point when $t = 0$.

Let $\vec{r}(t) = \langle t + \sin t, t - \cos t, te^{-t} \rangle$

Then $\vec{r}(0) = \langle 0, -1, 0 \rangle$

Also, $\vec{r}\,'(t) = \langle 1 + \cos t, 1 + \sin t, e^{-t} - te^{-t} \rangle$, so that

$\vec{r}\,'(0) = \langle 2, 1, 1 \rangle$

Therefore, the tangent line is given parametrically by

$x = 2t$, $y = -1 + t$, $z = t$

5. (16 points)

(a) **(6 points)** Find the cylindrical coordinates of the point P given by spherical coordinates $(4,\ 3\pi/4,\ 2\pi/3)$.

We have $\rho = 4$, $\theta = \frac{3\pi}{4}$, $\phi = \frac{2\pi}{3}$. From here

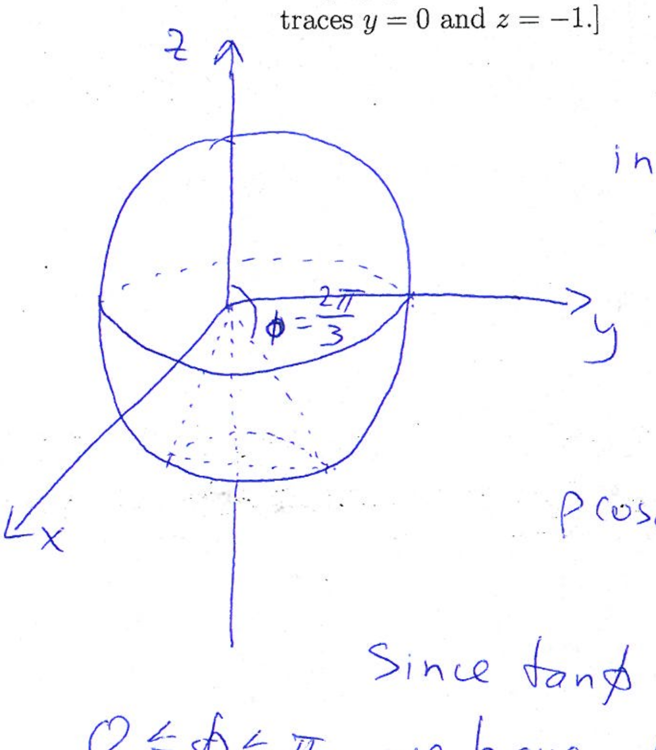
$r = \rho \sin\phi = (4)\sin\left(\frac{2\pi}{3}\right) = (4)\left(\frac{\sqrt{3}}{2}\right) = 2\sqrt{3}$,

$\theta = \frac{3\pi}{4}$,

and $z = \rho\cos\phi = (4)\cos\left(\frac{2\pi}{3}\right) = (4)\left(-\frac{1}{2}\right) = -2$.

The cylindrical coordinates of P are $\left(2\sqrt{3}, \frac{3\pi}{4}, -2\right)$.

(b) **(10 points)** The solid E lies strictly inside the sphere $x^2+y^2+z^2=4$ and strictly below the cone $z = -\sqrt{\frac{1}{3}(x^2+y^2)}$. Describe E using inequalities in spherical coordinates, and simplify your answer as much as possible. [Hint: you may find it helpful to consider the traces $y=0$ and $z=-1$.]



The equation of the sphere $x^2+y^2+z^2=4$ in spherical coordinates is $\rho^2 = 4$, and from here, $\rho = 2$.

Next, we convert the cone equation $z = -\sqrt{\frac{1}{3}(x^2+y^2)}$ to spherical coordinates:

$$\rho\cos\phi = -\sqrt{\frac{r^2}{3}} = -\frac{r}{\sqrt{3}} = -\frac{\rho\sin\phi}{\sqrt{3}}$$

$$\tan\phi = -\sqrt{3}$$

Since $\tan\phi$ is a negative number and we need $0 \le \phi \le \pi$, we have $\phi = \pi + \arctan(-\sqrt{3}) = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$.

From here, E is described in spherical coordinates by

$0 \le \rho \le 2$, $0 \le \theta \le 2\pi$, $\frac{2\pi}{3} \le \phi \le \pi$

6. **(17 points)** Let $\mathbf{r}(t) = \left\langle \frac{1}{2}e^t(\cos(t) + \sin(t)),\ \frac{1}{2}e^t(\cos(t) - \sin(t)) \right\rangle$.

(a) **(7 points)** What is the arclength of $\mathbf{r}(t)$ between $t = 0$ and $t = 1$?

$$\vec{r}\,'(t) = \left\langle \frac{1}{2}\left(e^t(\cos t + \sin t) + e^t(-\sin t + \cos t)\right), \frac{1}{2}\left(e^t(\cos t - \sin t) + e^t(-\sin t - \cos t)\right) \right\rangle$$

$$= \langle e^t \cos t, -e^t \sin t \rangle$$

$$\|\vec{r}\,'(t)\| = \sqrt{e^{2t}\cos^2 t + e^{2t}\sin^2 t} = \sqrt{e^{2t}(\cos^2 t + \sin^2 t)} = \sqrt{e^{2t}} = e^t$$

$$\text{Arc length} = \int_0^1 \|\vec{r}\,'(t)\|\,dt = \int_0^1 e^t\,dt = e^t\Big|_0^1 = e - 1$$

(b) **(7 points)** What is the curvature of $\mathbf{r}(t)$?

$$\vec{T}(t) = \frac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|} = \langle \cos t, -\sin t \rangle$$

$$\vec{T}\,'(t) = \langle -\sin t, -\cos t \rangle$$

$$\|\vec{T}\,'(t)\| = \sqrt{\sin^2 t + \cos^2 t} = 1$$

The curvature of $\vec{r}(t)$ is

$$K(t) = \frac{\|\vec{T}\,'(t)\|}{\|\vec{r}\,'(t)\|} = \frac{1}{e^t}$$

(c) **(3 points)** What is the radius of the osculating circle at $t = 1$?

The radius of the osculating circle at $t=1$ is

$$\frac{1}{\kappa(1)} = \frac{1}{1/e} = e$$