Solutions

MATH 2400: CALCULUS 3

7:30 - 10:00 am, Tues. Dec. 15, 2015

FINAL EXAM

I have neither given nor received aid on this exam.

Name:____ **Cancelled due to snowstorm and never given**

Check one below !

- ◯ **001** BULIN .... (9AM)
- ◯ **002** MOLCHO .... (10AM)
- ◯ **003** IH .... (11AM)
- ◯ **004** SPINA .... (12PM)
- ◯ **005** SPINA .... (1PM)
- ◯ **006** PRESTON .... (2PM)
- ◯ **007** PRESTON .... (3PM)
- ◯ **008** CHHAY .... (9AM)
- ◯ **009** WALTER .... (11AM)

If you have a question raise your hand and remain seated. In order to receive full credit your answer must be **complete**, **logical**, **legible**, and **correct**. Show all of your work, and give adequate explanations. No shown work (even with the correct final answer), no points! Only one answer to each problem! In case of two different answers to one problem, the lower score will be chosen!

In case of any need of scratch paper, use the backsides instead of extra sheets and in the problem(s) clearly indicate where your solutions are located.

DO NOT WRITE IN THIS BOX!

Problem	Points	Score
1	12 pts	
2	13 pts	
3	12 pts	
4	13 pts	
5	12 pts	
6	12 pts	
7	13 pts	
8	13 pts	
TOTAL	100 pts	

1. **(12 points)**

The following questions are true/false or multiple-choice. No partial credit will be given and no work is required to be shown on this problem only. Circle your answer.

(a) If f is a function, then $\operatorname{div}(\nabla f)$

(i) does not make sense (ii) makes sense and is always zero

(iii) makes sense and may be nonzero

(b) If $\mathbf{F}$ is a vector field, then $\operatorname{curl}(\operatorname{div}\mathbf{F})$

(i) does not make sense (ii) makes sense and is always zero

(iii) makes sense and may be nonzero

(c) If $\mathbf{F}$ is a vector field, then $\operatorname{div}(\operatorname{curl}\mathbf{F})$

(i) does not make sense (ii) makes sense and is always zero

(iii) makes sense and may be nonzero

If the components of $\vec{F}$ have continuous second-order partial derivatives, then $\operatorname{div}(\operatorname{curl}\vec{F})$ is always 0.

(d) If $f(x,y) = \frac{x^2-y^2}{x^2+y^2}$ for $(x,y) \neq (0,0)$, then defining $f(0,0) = 0$ makes f continuous.

(i) True (ii) False

(e) If $\mathbf{a}$ and $\mathbf{b}$ are nonzero vectors with $\mathbf{a} \times \mathbf{b} = \mathbf{0}$, then the angle between $\mathbf{a}$ and $\mathbf{b}$ is either 0 or π in radians.

(i) True (ii) False

(f) The osculating plane to a curve at a point is perpendicular to the vector

(i) Tangent $\mathbf{T}$ (ii) Normal $\mathbf{N}$ (iii) Binormal $\mathbf{B}$

2. **(13 points)** You are standing on a mountain with shape described by the equation $z = f(x, y) = 100 - 3x^2 - 2y^2$ at the point $(1, 2, 89)$.

(a) **(4 points)** In what unit direction should you move to descend as quickly as possible?

$\vec{\nabla} f = \langle -6x, -4y \rangle$

$\vec{\nabla} f(1,2) = \langle -6, -8 \rangle$

The unit direction is $\vec{u} = -\dfrac{\vec{\nabla} f(1,2)}{\|\vec{\nabla} f(1,2)\|} = \dfrac{\langle 6,8 \rangle}{10} = \langle \frac{3}{5}, \frac{4}{5} \rangle$

(b) **(3 points)** What is the rate of change of z in this direction from part (a)?

$D_{\vec{u}} f(1,2) = -\|\vec{\nabla} f(1,2)\| = -10$

2. (continued from previous page)

(c) **(4 points)** In what two unit directions can you move to stay at constant height $z = 89$?

First, we find a vector $\vec{v} = \langle v_1, v_2 \rangle$ such that $\vec{v} \cdot \vec{\nabla} f(1,2) = 0$. Since $\vec{\nabla} f(1,2) = \langle -6, -8 \rangle$, we have $-6v_1 - 8v_2 = 0$. We can pick $v_1 = -8$, $v_2 = 6$. Thus, we can pick $\vec{v} = \langle -8, 6 \rangle$. From here, the two unit directions are given by

$$\vec{w}_1 = \frac{\vec{v}}{\|\vec{v}\|} = \langle -\tfrac{4}{5}, \tfrac{3}{5} \rangle \text{ and}$$

$$\vec{w}_2 = -\frac{\vec{v}}{\|\vec{v}\|} = \langle \tfrac{4}{5}, \tfrac{-3}{5} \rangle$$

(d) **(2 points)** What is the rate of change in the directions from part (c)?

Since we are staying at constant height,

$$D_{\vec{w}_1} f(1,2) = D_{\vec{w}_2} f(1,2) = 0.$$

3. **(12 points)**

Consider the surface S given by $z = 2x^2 - 2x^2y + y - 1$.

(a) **(7 points)** Find the equation of the tangent plane at the point $(1, 1, 0)$.

$z_x = 4x - 4xy$, $z_x(1,1) = 0$

$z_y = -2x^2 + 1$, $z_y(1,1) = -1$

The equation of the tangent line is

$$z - 0 = 0(x-1) + (-1)(y-1)$$

$$z = -y + 1$$

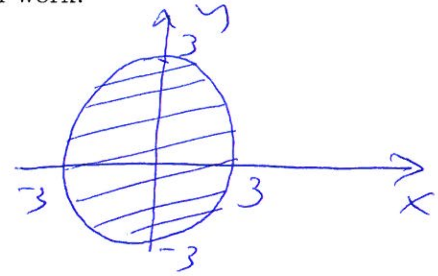
$$y + z = 1$$

(b) **(5 points)** Find the angle between the tangent plane from part (a) and the xy-plane.

A normal vector to the tangent plane is $\vec{n} = \langle 0,1,1 \rangle$. A normal vector to the xy-plane is the vector $\vec{k} = \langle 0,0,1 \rangle$. Therefore, the angle between the planes is given by

$$\arccos\left(\frac{\vec{n}\cdot\vec{k}}{\|\vec{n}\|\,\|\vec{k}\|}\right) = \arccos\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4}$$

4. **(13 points)** Find the point where the absolute (global) maximum of the function $f(x, y) = x^2 + y^2 - 2x$ occurs on the set $x^2 + y^2 \leq 9$. Show all work.



$f_x = 2x - 2 = 0 \Rightarrow x = 1$

$f_y = 2y = 0 \Rightarrow y = 0$

The critical point $(1,0)$ is in the set $x^2+y^2 \leq 9$.

Next, on the boundary $x^2+y^2=9$, we have $y^2 = 9-x^2$.

Thus, on the boundary,

$f(x,y) = x^2+y^2-2x = x^2+9-x^2-2x = 9-2x = f_1(x)$,

$-3 \leq x \leq 3$.

$f_1'(x) = -2 \neq 0$ for any x.

We still consider the endpoints $(-3,0)$, $(3,0)$.

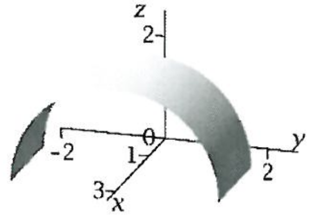
We have the following table:

(x,y)	$f(x,y)$
$(1,0)$	-1
$(-3,0)$	15
$(3,0)$	3

From the table above, the absolute maximum value is 15 and it occurs at the point $(-3,0)$.

5. **(12 points)**

Compute $\iint_S (z + xy)\,dS$, where S is the part of the cylinder $y^2 + z^2 = 4$ lying between the planes $x = 1$ and $x = 3$ and above the plane $z = 0$.



We parameterize S by

$$\vec{r}(x,\theta) = \langle x, 2\cos\theta, 2\sin\theta \rangle, \quad 1 \le x \le 3, \quad 0 \le \theta \le \pi$$

We have
$$\vec{r}_x \times \vec{r}_\theta = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 0 \\ 0 & -2\sin\theta & 2\cos\theta \end{vmatrix}$$

$$= \langle 0, -2\cos\theta, -2\sin\theta \rangle, \text{ and so}$$

$$\|\vec{r}_x \times \vec{r}_\theta\| = \sqrt{4\cos^2\theta + 4\sin^2\theta} = \sqrt{4} = 2$$

Therefore,

$$\iint_S (z+xy)\,dS = \int_1^3 \int_0^\pi (2\sin\theta + 2x\cos\theta)(2)\,d\theta\,dx$$

$$= 4\int_1^3 \left(\left(-\cos\theta + x\sin\theta\right)\Big|_{\theta=0}^{\pi} \right) dx = 4\int_1^3 2\,dx$$

$$= (4)(3-1)(2) = 16$$

6. **(12 points)** Let $\mathbf{F}(x,y,z) = \langle \sin y, x\cos y + z, y - \cos z \rangle$.

(a) Show that F is conservative by finding a potential function for it.

We need $f_x = \sin y$, $f_y = x\cos y + z$, $f_z = y - \cos z$.

$f_x = \sin y \Rightarrow f = \int \sin y \, dx = x \sin y + m(y,z)$

$f_y = x\cos y + m_y = x\cos y + z \Rightarrow m_y = z \Rightarrow m = \int z\,dy$
$= yz + k(z)$. Thus $f = x\sin y + yz + k(z)$.

$f_z = y + k'(z) = y - \cos z \Rightarrow k'(z) = -\cos z$
$\Rightarrow k = \int(-\cos z)\,dz = -\sin z + C$. We can take $C=0$.

Thus, a potential function for $\vec{F}$ is

$$f(x,y,z) = x\sin y + yz - \sin z$$

(b) Evaluate the line integral $\int_C \mathbf{F}\cdot d\mathbf{r}$ if C is the curve $\mathbf{r}(t) = \langle t, \pi t^2, \pi t^3\rangle$ for $0 \le t \le 1$.

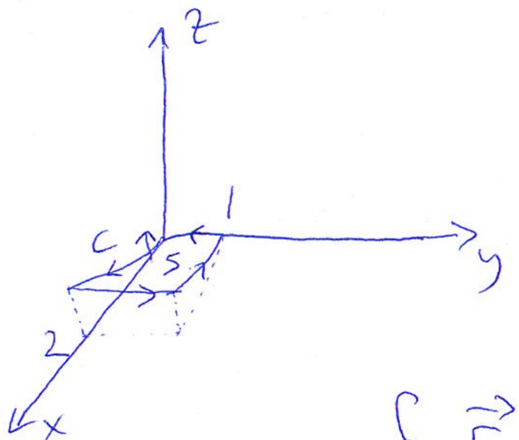
By The Fundamental Theorem for Line Integrals,

$$\int_C \vec{F}\cdot d\vec{r} = \int_C \vec{\nabla} f \cdot d\vec{r} = f(\vec{r}(1)) - f(\vec{r}(0))$$

$$= f(1,\pi,\pi) - f(0,0,0)$$

$$= \pi^2 - 0 = \pi^2$$

7. (13 points) Let S be the surface given by the function $z = x^2$, above the rectangle $0 \le x \le 2$, $0 \le y \le 1$. Let $\mathbf{F}(x,y,z) = \langle xy, y+z, 2z \rangle$. Compute the line integral $\oint_C \mathbf{F} \cdot d\mathbf{r}$ along the boundary curve C of S, oriented counterclockwise when looking down on it. (Hint: Use Stokes' Theorem.)



We orient S upward since then C has the induced positive orientation.

By Stokes' Theorem

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S \operatorname{curl} \vec{F} \cdot d\vec{S}$$

We have
$$\operatorname{curl} \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & y+z & 2z \end{vmatrix} = \langle -1, 0, -x \rangle$$

We parameterize S by $\vec{r}(x,y) = \langle x, y, x^2 \rangle$, $0 \le x \le 2$, $0 \le y \le 1$. We have
$$\vec{r}_x \times \vec{r}_y = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 2x \\ 0 & 1 & 0 \end{vmatrix} = \langle -2x, 0, 1 \rangle.$$

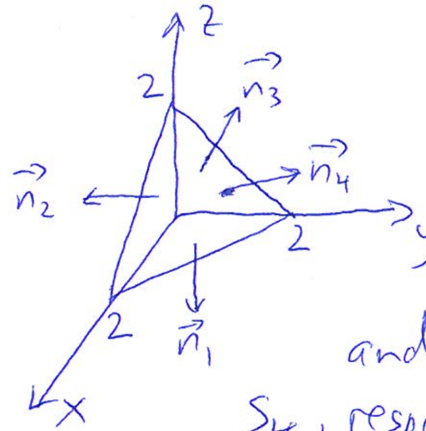
Therefore,

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S \operatorname{curl} \vec{F} \cdot d\vec{S} = \int_0^1 \int_0^2 \langle -1, 0, -x \rangle \cdot \langle -2x, 0, 1 \rangle \, dx\, dy$$

$$= \int_0^1 \int_0^2 x \, dx\, dy = \int_0^1 \left(\frac{x^2}{2} \Big|_{x=0}^{2} \right) dy = \int_0^1 2\, dy = 2$$

8. **(13 points)** Let E be the region in the first octant bounded by the planes $x = 0$, $y = 0$, $z = 0$, and $x + y + z = 2$, and let S be its boundary surface with outward orientation.

(a) **(4 points)** Draw a picture of E and S, and write the unit normals for each of the four surfaces.



Let S_1, S_2, and S_3 be the parts of S that lie on the xy-plane, xz-plane, and yz-plane, respectively. Let S_4 be the part of the plane $x+y+z=2$. Let $\vec{n}_1, \vec{n}_2, \vec{n}_3$, and $\vec{n}_4$ be the unit normals for S_1, S_2, S_3, and S_4, respectively. Then, the unit normals are given by $\vec{n}_1 = -\vec{k}$, $\vec{n}_2 = -\vec{j}$, $\vec{n}_3 = -\vec{i}$, and $\vec{n}_4 = \frac{1}{\sqrt{3}}\vec{i} + \frac{1}{\sqrt{3}}\vec{j} + \frac{1}{\sqrt{3}}\vec{k}$.

(b) **(5 points)** If $\mathbf{F} = \langle x, -y, z\rangle$, compute the flux $\iint_S \mathbf{F}\cdot\mathbf{n}\,dS$ directly. (Hint: some terms are zero.)

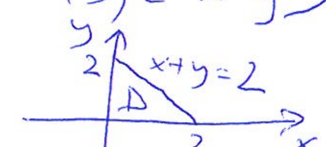
$$\iint_S \vec{F}\cdot\vec{n}\,dS = \iint_{S_1} \vec{F}\cdot\vec{n}\,dS + \iint_{S_2} \vec{F}\cdot\vec{n}\,dS + \iint_{S_3} \vec{F}\cdot\vec{n}\,dS + \iint_{S_4} \vec{F}\cdot\vec{n}\,dS$$

$$\iint_{S_1} \vec{F}\cdot\vec{n}\,dS = \iint_{S_1} \langle x,-y,0\rangle\cdot\langle 0,0,-1\rangle\,dS = \iint_{S_1} 0\,dS = 0$$

$$\iint_{S_2} \vec{F}\cdot\vec{n}\,dS = \iint_{S_2} \langle x,0,z\rangle\cdot\langle 0,-1,0\rangle\,dS = \iint_{S_2} 0\,dS = 0$$

$$\iint_{S_3} \vec{F}\cdot\vec{n}\,dS = \iint_{S_3} \langle 0,-y,z\rangle\cdot\langle -1,0,0\rangle\,dS = \iint_{S_3} 0\,dS = 0$$

We parameterize S_4 by $\vec{r}(x,y) = \langle x, y, 2-x-y\rangle$, $(x,y)\in D$, where D is the triangular region



Then $\vec{r}_x \times \vec{r}_y = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{vmatrix} = \langle 1,1,1\rangle$ and so $\iint_{S_4} \vec{F}\cdot d\vec{S}$

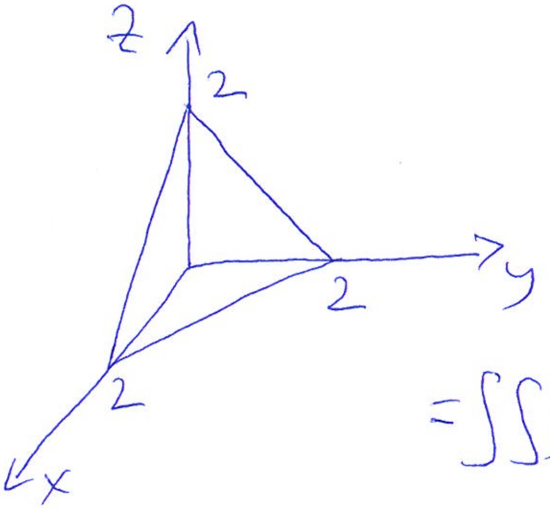
$= \iint_D \langle x,-y,2-x-y\rangle\cdot\langle 1,1,1\rangle\,dy\,dx = \int_0^2\int_0^{2-x} (2-2y)\,dy\,dx$

$= \int_0^2 \left(\left(2y-y^2\right)\Big|_{y=0}^{2-x}\right)dx = \int_0^2 \left(2(2-x)-(2-x)^2\right)dx = \int_0^2 (2-x)(2-(2-x))\,dx = \int_0^2 (2-x)x\,dx$

$= \int_0^2 (2x-x^2)\,dx = \left(x^2 - \frac{x^3}{3}\right)\Big|_0^2 = 4 - \frac{8}{3} = \frac{4}{3}$. Hence $\iint_S \vec{F}\cdot\vec{n}\,dS = \frac{4}{3}$.

8. (continued from previous page)

(c) **(4 points)** Compute the same flux from part (b) using the Divergence Theorem.



By the Divergence Theorem,

$$\iint_S \vec{F}\cdot\vec{n}\,dS = \iiint_E \operatorname{div}\vec{F}\,dV$$

$$= \iiint_E (1-1+1)\,dV = \iiint_E dV$$

$$= \int_0^2\int_0^{2-x}\int_0^{2-x-y} dz\,dy\,dx = \int_0^2\int_0^{2-x}(2-x-y)\,dy\,dx$$

$$= \int_0^2\left(\left((2-x)y-\frac{y^2}{2}\right)\Big|_{y=0}^{2-x}\right)dx = \int_0^2\left((2-x)^2-\frac{(2-x)^2}{2}\right)dx$$

$$= \int_0^2 \frac{(2-x)^2}{2}\,dx = -\frac{(2-x)^3}{6}\Big|_0^2 = \frac{8}{6} = \frac{4}{3}$$

(Alternatively, $\iiint_E dV = \text{Volume}(E)$

$= \left(\frac{1}{3}\right)(\text{Area of base})(\text{Height})$

$= \left(\frac{1}{3}\right)\left(\frac{(2)(2)}{2}\right)(2) = \frac{4}{3}$)