MATH 2400: CALCULUS 3

5:15 - 6:45 pm, Mon. Nov. 16, 2015

MIDTERM 3

I have neither given nor received aid on this exam.

Name:____

Check one below !

- ◯ **001** Bulin .... (9am)
- ◯ **002** Molcho .... (10am)
- ◯ **003** Ih .... (11am)
- ◯ **004** Spina .... (12pm)
- ◯ **005** Spina .... (1pm)
- ◯ **006** Preston .... (2pm)
- ◯ **007** Preston .... (3pm)
- ◯ **008** Chhay .... (9am)
- ◯ **009** Walter .... (11am)

If you have a question raise your hand and remain seated. In order to receive full credit your answer must be **complete**, **logical**, **legible**, and **correct**. Show all of your work, and give adequate explanations. No shown work even with the correct final answer, no points ! Only one answer to each problem ! In case of two different answers to one problem, the lower score will be chosen !

In case of any need of scratch paper, use the backsides instead of extra sheets and in the problem(s) clearly indicate where your solutions are located.

DO NOT WRITE IN THIS BOX!

Problem	Points	Score
1	16 pts	
2	16 pts	
3	17 pts	
4	17 pts	
5	17 pts	
6	17 pts	
TOTAL	100 pts	

1. **(16 points)** Let E be the solid bounded by the planes

$$x = 0; \quad y = 0; \quad z = 0, \quad \text{and} \quad 3x + 2y + z = 6.$$

Suppose that it has density function $\rho(x, y, z) = x$. Then find the mass of E.

Solution. The intersection of E and the xy-plane is the region on the xy-plane bounded by

$$x = 0; \quad y = 0; \quad 3x + 2y = 6.$$

(Drawing this region as well as E will be useful here.)

So we have total mass

$$\begin{aligned}
m &= \int\int\int_E x \, dV \\
&= \int_0^2 \int_0^{-\frac{3}{2}x+3} \int_0^{-3x-2y+6} x \, dz \, dy \, dx \\
&= \int_0^2 \left(x \int_0^{-\frac{3}{2}x+3} (-3x - 2y + 6) \, dy \right) dx \\
&= \int_0^2 x[-3xy - y^2 + 6y]_{y=0}^{y=-\frac{3}{2}x+3} \, dx \\
&= \int_0^2 x \left(\frac{9}{4}x^2 - 9x + 9 \right) dx \\
&= \int_0^2 \left(\frac{9}{4}x^3 - 9x^2 + 9x \right) dx \\
&= \left[\frac{9}{16}x^4 - 3x^3 + \frac{9}{2}x^2 \right]_0^2 \\
&= 9 - 24 + 18 \\
&= 3.
\end{aligned}$$

2. (16 points) Let C be the wire given by the helix

$$x = 2\cos t; \quad y = 2\sin t; \quad z = 3t,$$

where $0 \leq t \leq 2\pi$. Suppose that it has density function $\rho(x, y, z) = z$. Then find $\overline{z}$ (= the z-coordinate of the center of mass of C).

Solution. First, we have the total mass equal to

$$\begin{aligned}
m &= \int_C z \, ds \\
&= \int_0^{2\pi} 3t\sqrt{(-2\sin t)^2 + (2\cos t)^2 + 3^2} \, dt \\
&= 3\sqrt{13} \int_0^{2\pi} t \, dt \\
&= \frac{3\sqrt{13}}{2}[t^2]_0^{2\pi} \\
&= 6\sqrt{13}\pi^2.
\end{aligned}$$

Second, we have

$$\begin{aligned}
\int_C z \cdot z \, ds &= \int_C z^2 \, ds \\
&= \int_0^{2\pi} 9t^2\sqrt{(-2\sin t)^2 + (2\cos t)^2 + 3^2} \, dt \\
&= 9\sqrt{13} \int_0^{2\pi} t^2 \, dt \\
&= 3\sqrt{13}[t^3]_0^{2\pi} \\
&= 24\sqrt{13}\pi^3.
\end{aligned}$$

Therefore we have

$$\begin{aligned}
\overline{z} &= \frac{24\sqrt{13}\pi^3}{6\sqrt{13}\pi^2} \\
&= 4\pi.
\end{aligned}$$

3. (17 points) Compute the surface area of the part of the paraboloid

$$z = 16 - x^2 - y^2$$

that lies above the xy-plane.

Solution. (Drawing the graph is useful here.)

We use polar coordinates

$$x = u\cos v \quad \text{and} \quad y = u \sin v \quad (0 \le u \le 4 \quad \text{and} \quad 0 \le v \le 2\pi)$$

to parametrize the surface by

$$\mathbf{r}(u,v) = \langle u\cos v,\ u\sin v,\ 16 - u^2\rangle.$$

So we have

$$\mathbf{r}_u = \langle \cos v,\ \sin v,\ -2u\rangle \quad \text{and} \quad \mathbf{r}_v = \langle -u\sin v,\ u\cos v,\ 0\rangle.$$

Thus it follows that

$$\mathbf{r}_u \times \mathbf{r}_v = \langle 2u^2\cos v,\ 2u^2\sin v,\ u\rangle.$$

(You need to show detailed work here.)

Therefore we have

$$|\mathbf{r}_u \times \mathbf{r}_v| = \sqrt{(2u^2\cos v)^2 + (2u^2\sin v)^2 + u^2} = u\sqrt{4u^2+1}$$

(note $u \ge 0$) and the surface area

$$\begin{aligned}
\int_0^{2\pi}\int_0^4 u\sqrt{4u^2+1}\,du\,dv &= \frac{2}{3}\cdot\frac{1}{8}\int_0^{2\pi}[(4u^2+1)^{\frac{3}{2}}]_{u=0}^{u=4}\,dv \qquad (\text{note } d(4u^2+1) = 8u\,du)\\
&= \frac{2}{3}\cdot\frac{1}{8}\cdot 2\pi\cdot(65^{\frac{3}{2}} - 1)\\
&= \frac{\pi}{6}(65^{\frac{3}{2}} - 1).
\end{aligned}$$

(The first equality in this integral calculation can be done explicitly by $w = 4u^2 + 1$ and $dw = 8u\,du$, too.)

4. (17 points) Consider the vector field

$$\mathbf{F}(x, y) = (3 + 2xy)\mathbf{i} + (x^2 - 3y^2)\mathbf{j}.$$

(a) If it is conservative, then find all (real-valued) functions $f(x, y)$ such that $\mathbf{F} = \nabla f$. Otherwise, tell why not.

Solution. (You could show that $\mathbf{F} = \nabla \times \mathbf{F} = \mathbf{0}$. But here you do not have to.)

Let us find $f(x, y)$ such that $\mathbf{F} = \nabla f$. Observe

$$\frac{\partial f}{\partial x} = 3 + 2xy \;\Rightarrow\; f(x, y) = \int (3 + 2xy)\; dx = 3x + x^2 y + g(y),$$

where $g(y)$ is a function depending on y, but not on x. So we have

$$x^2 + g'(y) = \frac{\partial f}{\partial y} = x^2 - 3y^2 \;\Rightarrow\; g'(y) = -3y^2 \;\Rightarrow\; g(y) = -y^3 + K,$$

where $g'(y) = \frac{dg(y)}{dy}$ and K is a constant independent of x and y.

Therefore it follows that $\mathbf{F}$ is conservative and that $f(x, y) = 3x + x^2 y - y^3 + K$, where K is a constant (independent of x and y).

(b) Suppose that C is the closed curve on the xy-plane that is formed by the square with vertices $(0, 0)$, $(1, 0)$, $(1, 1)$, and $(0, 1)$ and that is traversed in that order. Then evaluate

$$\int_C \mathbf{F} \cdot d\mathbf{r},$$

where $\mathbf{r}(t)$ is a vector function giving C.

Solution. Since C is closed and since we know from (a) that $\mathbf{F}$ is conservative, we immediately conclude that $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$.

5. (17 points) Consider the triangle C consisting of the line segments from $(0,0)$ to $(1,0)$, from $(1,0)$ to $(0,1)$, and from $(0,1)$ to $(0,0)$. Evaluate

$$\int_C x\ dx + xy\ dy$$

in two ways: (a) directly and (b) via Green's theorem.

(a) Directly

Solution. We need to parameterize C explicitly. Here is one way. (Drawing the triangle C is useful.)

First, on $C_1 : (0,0) \to (1,0)$ use x itself ($0 \le x \le 1$) and $y = 0$;

second, on $C_2 : (1,0) \to (0,1)$ use $x = -y+1$ and y itself ($0 \le y \le 1$); and

third, on $C_3 : (0,1) \to (0,0)$ use $x = 0$ and $-y+1$ itself ($0 \le y \le 1$).

Now we have

$$\begin{aligned}
\int_{C_1} x\ dx + xy\ dy &= \int_0^1 x\ dx = \frac{1}{2}[x^2]_0^1 = \frac{1}{2}; \\
\int_{C_2} x\ dx + xy\ dy &= \int_0^1 (-y+1)(-1)\ dy + (-y+1)y\ dy \\
&= \int_0^1 (-y^2+2y-1)\ dy = \left[-\frac{1}{3}y^3 + y^2 - y\right]_0^1 \\
&= -\frac{1}{3} + 1 - 1 = -\frac{1}{3}; \quad \text{and} \\
\int_{C_3} x\ dx + xy\ dy &= \int_0^1 0\ dx + 0\ dy = 0.
\end{aligned}$$

Therefore it follows that

$$\begin{aligned}
\int_C x\ dx + xy\ dy &= \sum_{i=1}^{3}\left(\int_{C_i} x\ dx + xy\ dy\right) \\
&= \frac{1}{2} - \frac{1}{3} + 0 = \frac{1}{6}.
\end{aligned}$$

(b) Via Green's theorem

Solution. Let D be the solid triangle enclosed by the 3 vertices. Since $\frac{\partial(xy)}{\partial x} = y$ and $\frac{\partial x}{\partial y} = 0$, Green's theorem implies that

$$\begin{aligned}
\int_C x\ dx + xy\ dy &= \int\int_D y\ dA = \int_0^1 \int_0^{-x+1} y\ dy\ dx = \frac{1}{2}\int_0^1 [y^2]_{y=0}^{y=-x+1}\ dx \\
&= \frac{1}{2}\int_0^1 (x^2 - 2x + 1)\ dx = \frac{1}{2}\left[\frac{1}{3}x^3 - x^2 + x\right]_0^1 = \frac{1}{2}\left(\frac{1}{3} - 1 + 1\right) = \frac{1}{6}.
\end{aligned}$$

6. (17 points) Let R be the region on the xy-plane bounded by the following curves

$$xy = 1; \quad xy = 3; \quad \frac{y}{x^2} = 2; \quad \frac{y}{x^2} = 5.$$

Find the area of R. (*Hint.* A change of variables may be useful.)

Solution. (Drawing the region R will be helpful.)

Let us use the change of variables

$$u = xy \qquad \text{and} \qquad v = \frac{y}{x^2}.$$

Then we have $1 \le u \le 3$ and $2 \le v \le 5$. Solve the simultaneous equations for x and y to get

$$x = \sqrt[3]{\frac{u}{v}} \; (= u^{\frac{1}{3}} v^{-\frac{1}{3}}) \qquad \text{and} \qquad y = \sqrt[3]{u^2 v} \; (= u^{\frac{2}{3}} v^{\frac{1}{3}}).$$

(You need to show detailed work.) It follows that

$$\frac{\partial x}{\partial u} = \frac{1}{3} \cdot \frac{1}{\sqrt[3]{u^2 v}} \; (= \frac{1}{3} u^{-\frac{2}{3}} v^{-\frac{1}{3}}); \quad \frac{\partial x}{\partial v} = -\frac{1}{3} \cdot \frac{1}{v} \cdot \sqrt[3]{\frac{u}{v}} \; (= -\frac{1}{3} u^{\frac{1}{3}} v^{-\frac{4}{3}});$$

$$\frac{\partial y}{\partial u} = \frac{2}{3} \cdot \sqrt[3]{\frac{v}{u}} \; (= \frac{2}{3} u^{-\frac{1}{3}} v^{\frac{1}{3}}); \qquad \frac{\partial y}{\partial v} = \frac{1}{3} \cdot \sqrt[3]{\left(\frac{u}{v}\right)^2} \; (= \frac{1}{3} u^{\frac{2}{3}} v^{-\frac{2}{3}}).$$

(You need to show detailed work.)

Hence the Jacobian is equal to

$$\frac{\partial(x, y)}{\partial(u, v)} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u} = \frac{1}{9} \cdot \frac{1}{v} + \frac{2}{9} \cdot \frac{1}{v} = \frac{1}{3} \cdot \frac{1}{v} \; (> 0).$$

(Alternatively, you could use implicit differentiation to find the 4 partial derivatives and this Jacobian, too.)

Therefore (drawing the regions in the uv-plane and the xy-plane is useful here and) the area of R is equal to

$$\begin{aligned} \int\!\!\int_R dA &= \int_2^5 \int_1^3 \frac{1}{3} \cdot \frac{1}{v} \, du \, dv \\ &= \frac{2}{3} \int_2^5 \frac{1}{v} \, dv \\ &= \frac{2}{3} [\ln v]_2^5 && \text{(since } v > 0\text{)} \\ &= \frac{2}{3} (\ln 5 - \ln 2). \end{aligned}$$