# MATH 2400: CALCULUS 3

**5:15 - 6:45 pm, Mon. Oct. 19, 2015**

# MIDTERM 2

I have neither given nor received aid on this exam.

**Name:**________________________________

Check one below !

- ◯ **001** BULIN ................. (9AM)
- ◯ **002** MOLCHO .............. (10AM)
- ◯ **003** IH .................... (11AM)
- ◯ **004** SPINA ................. (12PM)
- ◯ **005** SPINA ................. (1PM)
- ◯ **006** PRESTON ............... (2PM)
- ◯ **007** PRESTON ............... (3PM)
- ◯ **008** CHHAY ................. (9AM)
- ◯ **009** WALTER .............. (11AM)

If you have a question raise your hand and remain seated. In order to receive full credit your answer must be **complete**, **logical**, **legible**, and **correct**. Show all of your work, and give adequate explanations. No shown work even with the correct final answer, no points ! Only one answer to each problem ! In case of two different answers to one problem, the lower score will be chosen !

**DO NOT WRITE IN THIS BOX!**

| Problem | Points | Score |
|---|---|---|
| 1 | 16 pts | |
| 2 | 17 pts | |
| 3 | 17 pts | |
| 4 | 17 pts | |
| 5 | 16 pts | |
| 6 | 17 pts | |
| TOTAL | 100 pts | |

**1. (16 points)** Suppose that

$$f(x, y) = x^3 + 6x^2y + axy^2 + by^3,$$

for some constants $a$ and $b$. Then find $a$ and $b$ such that

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$$

for every $(x, y)$.

(Note that this equation can also be equivalently written as $f_{xx} + f_{yy} = 0$.)

**Solution.** Look at

$$\frac{\partial f}{\partial x} = 3x^2 + 12xy + ay^2 \quad \text{and} \quad \frac{\partial^2 f}{\partial x^2} = 6x + 12y;$$
$$\frac{\partial f}{\partial y} = 6x^2 + 2axy + 3by^2 \quad \text{and} \quad \frac{\partial^2 f}{\partial y^2} = 2ax + 6by.$$

So

$$0 = (6x + 12y) + (2ax + 6by) = 2(3 + a)x + 6(2 + b)y$$

for all $x$ and $y$.

Therefore we have $a = -3$ and $b = -2$.

**2. (17 points)** Consider the hyperbolic paraboloid surface given by the equation

$$z = 2x^2 - 3y^2.$$

(a) **(12 points)** In what (unit) direction does $z$ have its maximum rate of change at the point $(2, 1)$ ?

**Solution.** Let $f(x, y) = 2x^2 - 3y^2$. Then we have $\nabla f(x, y) = \langle 4x, -6y \rangle$ and $\nabla f(2, 1) = \langle 8, -6 \rangle$. This vector has length $\sqrt{8^2 + (-6)^2} = 10$. So the desired unit direction is $\langle \frac{8}{10}, \frac{-6}{10} \rangle = \langle \frac{4}{5}, \frac{-3}{5} \rangle$.

(b) **(5 points)** What is the maximum rate of change in the direction in (a) ?

**Solution.** It is equal to $|\nabla f(2, 1)| = \sqrt{8^2 + (-6)^2} = 10$.

**3. (17 points)** Find and classify the critical points (local maxima, local minima, or saddle points) of

$$f(x,y) = x^3 + y^3 - 3xy.$$

**Solution.** Look at

$$\frac{\partial f}{\partial x} = 3x^2 - 3y, \qquad \frac{\partial^2 f}{\partial x^2} = 6x, \qquad \frac{\partial^2 f}{\partial y \partial x} = -3;$$

$$\frac{\partial f}{\partial y} = 3y^2 - 3x, \qquad \frac{\partial^2 f}{\partial y^2} = 6y.$$

To find the critical points, solve

$$3x^2 - 3y = 3y^2 - 3x = 0;$$

$y = x^2$ and $0 = x^4 - x = x(x^3 - 1) = x(x-1)(x^2 + x + 1)$; and get $(x,y) = (0,0)$ or $(1,1)$.

Note

$$D(x,y) = (6x)(6y) - (-3)^2 = 36xy - 9.$$

Thus we have

$$D(0,0) = -9 < 0, \qquad D(1,1) = 27 > 0, \qquad \text{and} \qquad f_{xx}(1,1) = 6 > 0.$$

Therefore $f$ has local minimum $f(1,1) = -2$ at $(1,1)$ and a saddle point at $(0,0)$.

**4. (17 points)** Find the tangent plane to the surface defined by the equation

$$x^2 z + yz = 1$$

at the point $(1, 1, \frac{1}{2})$.

**Solution.** Use implicit differentiation to find

$$2xz + x^2 \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial x} = 0 \quad \text{or} \quad \frac{\partial z}{\partial x} = -\frac{2xz}{x^2 + y}$$

$$x^2 \frac{\partial z}{\partial y} + z + y \frac{\partial z}{\partial y} = 0 \quad \text{or} \quad \frac{\partial z}{\partial y} = -\frac{z}{x^2 + y},$$

where $x^2 + y \neq 0$. Evaluate these two partial derivatives at $(1, 1, \frac{1}{2})$ to get $-\frac{1}{2}$ and $-\frac{1}{4}$, respectively. So the desired tangent plane is given by

$$z - \frac{1}{2} = -\frac{1}{2}(x - 1) - \frac{1}{4}(y - 1) \quad \text{or} \quad 2x + y + 4z - 5 = 0.$$

(Alternatively, you could use $z = \frac{1}{x^2+y}$ to find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ and evaluate these at $(x, y) = (1, 1)$ above.)

**5. (16 points)** Let

$$z = f(x,y), \quad x = u^2 - v^3, \quad y = u + 2v^2.$$

Suppose that $f$ is a differentiable function of $x$ and $y$, and that

$$\left.\frac{\partial z}{\partial x}\right|_{(x,y)=(-7,9)} = -2 \quad \text{and} \quad \left.\frac{\partial z}{\partial y}\right|_{(x,y)=(-7,9)} = 3.$$

Then find

$$\left.\frac{\partial z}{\partial v}\right|_{(u,v)=(1,2)}.$$

(Note that, for example, $\left.\frac{\partial z}{\partial x}\right|_{(x,y)=(-7,9)}$ (respectively $\left.\frac{\partial z}{\partial v}\right|_{(u,v)=(1,2)}$) means the value of $\frac{\partial z}{\partial x}$ at $(x,y) = (-7,9)$ (respectively the value of $\frac{\partial z}{\partial v}$ at $(u,v) = (1,2)$).)

**Solution.** Use the chain rule to find

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}. \tag{1}$$

Also note that

$$\frac{\partial x}{\partial v} = -3v^2 \quad \text{and} \quad \frac{\partial y}{\partial v} = 4v; \quad \text{so}$$

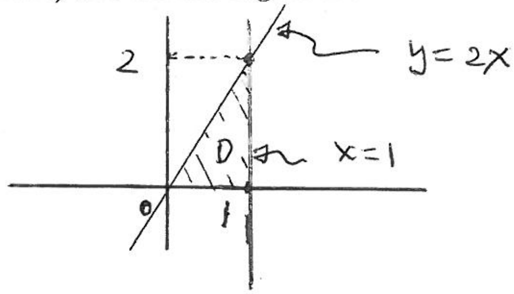
$$\left.\frac{\partial x}{\partial v}\right|_{(u,v)=(1,2)} = -12 \quad \text{and} \quad \left.\frac{\partial y}{\partial v}\right|_{(u,v)=(1,2)} = 8. \tag{2}$$

Look at the formulas for $x$ and $y$ in terms of $u$ and $v$ and note that $(u,v) = (1,2)$ implies $(x,y) = (-7,9)$. Then use the results in (2) and the given data in the problem to evaluate the partial derivatives in (1) at $(u,v) = (1,2)$ and get

$$\left.\frac{\partial z}{\partial v}\right|_{(u,v)=(1,2)} = (-2)(-12) + 3 \cdot 8 = 48.$$

**6. (17 points)** Let $D$ be the region on the $xy$-plane that is bounded by the $x$-axis, the vertical line $x = 1$, and the line $y = 2x$.

(a) **(3 points)** Sketch the region $D$.




(b) **(14 points)** Find the double integral of $\sqrt{1-x^2}$ over $D$.

**Solution.** From (a) above, we have

$$\begin{aligned}
\iint_D \sqrt{1-x^2}\, dA &= \int_0^1 \int_0^{2x} \sqrt{1-x^2}\, dy\, dx \\
&= \int_0^1 \sqrt{1-x^2} \left( \int_0^{2x} dy \right) dx \\
&= \int_0^1 \sqrt{1-x^2} \cdot 2x\, dx \\
&= -\int_0^1 \sqrt{1-x^2} \cdot (-2x)\, dx \\
&= -\int_0^1 (1-x^2)^{\frac{1}{2}} \cdot (-2x)\, dx \qquad \text{(note } (1-x^2)' = -2x\text{)} \\
&= -\frac{2}{3}[(1-x^2)^{\frac{3}{2}}]_0^1 \\
&= -\frac{2}{3}(0-1) \\
&= \frac{2}{3}.
\end{aligned}$$

(Alternatively, you could explicitly use $u = 1-x^2$; $du = -2x\, dx$ to find $\int \sqrt{1-x^2} \cdot 2x\, dx$ above.)