MATH 2400: CALCULUS 3

5:15 - 6:45 pm, Mon. Sep. 21, 2015

MIDTERM 1

I have neither given nor received aid on this exam.

Name:______________________________

Check one below !

- ◯ **001** Bulin .... (9AM)
- ◯ **002** Molcho .... (10AM)
- ◯ **003** Ih .... (11AM)
- ◯ **004** Spina .... (12PM)
- ◯ **005** Spina .... (1PM)
- ◯ **006** Preston .... (2PM)
- ◯ **007** Preston .... (3PM)
- ◯ **008** Chhay .... (9AM)
- ◯ **009** Walter .... (11AM)

If you have a question raise your hand and remain seated. In order to receive full credit your answer must be **complete**, **logical**, **legible**, and **correct**. Show all of your work, and give adequate explanations. No shown work even with the correct final answer, no points ! Only one answer to each problem ! In case of two different answers to one problem, the lower score will be chosen !

DO NOT WRITE IN THIS BOX!

Problem	Points	Score
1	17 pts	
2	17 pts	
3	16 pts	
4	17 pts	
5	16 pts	
6	17 pts	
TOTAL	100 pts	

1. **(17 points)** Let P_0 be the point $(1, 1, 2)$ and let $\wp$ be the plane given by the equation $2x - y + 2z = 2$

   (a) **(9 points)** Find parametric equations of the line L passing through the point P_0 and perpendicular to the plane $\wp$.

   Solution: Since $\wp \perp \langle 2, -1, 2\rangle$, it follows that $L // \langle 2, -1, 2\rangle$. Thus L has parametric equation

   $$x = 1 + 2t; y = 1 - t; z = 2 + 2t.$$

   (b) **(8 points)** Find the intersection point of the line L in (a) above and the plane $\wp$.

   Solution: We need to solve the equation

   $$2(1 + 2t) - (1 - t) + 2(2 + 2t) = 2.$$

   It gives $9t = -3$ or

   $$t = -1/3.$$

   Hence the desired intersection point is $(1/3, 4/3, 4/3)$.

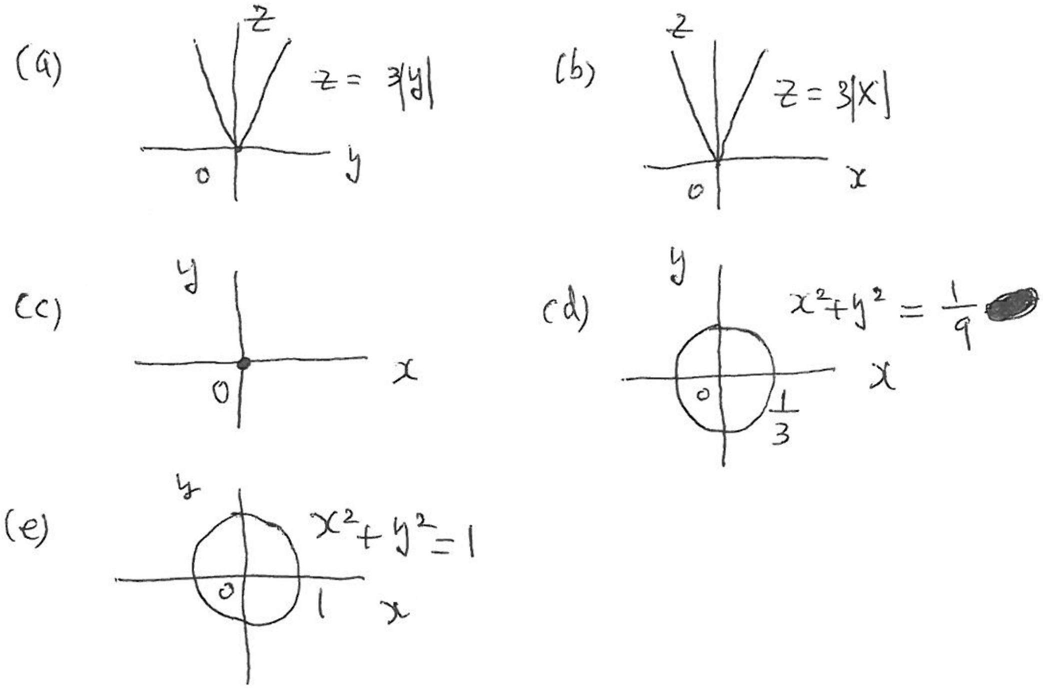
2. (17 points) Consider the surface S given by the equation

$$z = 3\sqrt{x^2 + y^2}.$$

(a) **(10 points)** Sketch the intersections of the surface S with each of the five planes

(a) $x = 0$; (b) $y = 0$; (c) $z = 0$; (d) $z = 1$; (e) $z = 3$.

What does each of these intersections look like roughly on each plane ?



(b) **(7 points)** Write down the equation of the cone with apex at $(0, 0, 0)$, which is symmetric about the z-axis and which intersects the plane given by the equation $z = 1$ at a circle of radius 2.

Solution: The cone has the equations $x^2 + y^2 = 4$ and $z = 1$ on the plane defined by $z = 1$. So the cone itself is given by the equation $x^2 + y^2 = 4z^2$.

3. **(16 points)** Which of the following is the angle between the (big) diagonal of a unit cube and one of its edges, where the diagonal and the edge start at the same point ? (Circle one of them and justify your answer. Show all work for full credit.)

(a) $\arcsin \frac{1}{\sqrt{3}}$ (b) $\arccos \frac{1}{\sqrt{3}}$ (c) $\arcsin \frac{2}{\sqrt{6}}$ (d) $\arccos \frac{2}{\sqrt{6}}$

Solution: We may assume that the cube has base with vertices $(0,0,0)$, $(1,0,0)$, $(1,1,0)$, $(0,1,0)$ on the xy-plane and the other four vertices $(0,0,1)$, $(1,0,1)$, $(1,1,1)$, $(0,1,1)$. We take the big diagonal to be

$$\vec{a} = \overrightarrow{(0,0,0),(1,1,1)} = \langle 1,1,1 \rangle$$

and the edge to be

$$\vec{b} = \overrightarrow{(0,0,0),(1,0,0)} = \langle 1,0,0 \rangle.$$

Suppose that θ is the angle between $\vec{a}$ and $\vec{b}$. Then we have

$$\cos\theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}| \cdot |\vec{b}|} = \frac{1}{\sqrt{3} \cdot 1} = \frac{1}{\sqrt{3}}.$$

Thus we have $\theta = \arccos \frac{1}{\sqrt{3}}$, namely (b).

4. **(17 points)** Let C be the helix $\boldsymbol{r}(t) = \langle \sin(\pi t), \cos(\pi t), t \rangle$ and let S be the sphere $x^2+y^2+z^2 = 5$.

(a) **(8 points)** At what points do the helix C intersect the sphere S ?

Solution: We need to solve the equation

$$\sin^2(\pi t) + \cos^2(\pi t) + t^2 = 5.$$

This equation implies

$$t^2 = 4$$

since $\sin^2(\pi t) + \cos^2(\pi t) = 1$. So we have $t = \pm 2$ and it follows that the desired intersection points are $(\sin(\pm 2\pi), \cos(\pm 2\pi), \pm 2)$, namely

$$(0, 1, \pm 2).$$

(b) **(9 points)** Find the tangent line to the helix C at the intersection point having positive z-coordinate.

Solution: The intersection point with positive z-coordinate is $(0, 1, 2)$, namely the point coming from $t = 2$. The desired tangent line passes through $(0, 1, 2)$ and has directional vector $\boldsymbol{r}'(2)$. Look at

$$\boldsymbol{r}'(t) = \langle \pi \cos(\pi t), -\pi \sin(\pi t), 1 \rangle$$

and get

$$\boldsymbol{r}'(2) = \langle \pi \cos(2\pi), -\pi \sin(2\pi), 1 \rangle = \langle \pi, 0, 1 \rangle.$$

Therefore the tangent line is given by

$$x = 0 + \pi t = \pi t; y = 1 + 0t = 1; z = 2 + 1 \cdot t = 2 + t.$$

5. **(16 points)**

(a) **(8 points)** Find the spherical coordinates of the point given by $(1, 1, -\sqrt{2})$ in rectangular coordinates.

Solution: Recall the general formulae for $(x, y, z) \leftrightarrow (\rho, \theta, \phi)$

$$x = \rho \sin \phi \cos \theta; y = \rho \sin \phi \sin \theta; z = \rho \cos \phi.$$

Now look at

$$\rho = \sqrt{1^1 + 1^2 + (-\sqrt{2})^2} = \sqrt{4} = 2;$$

$$\cos \phi = \frac{-\sqrt{2}}{2}, \quad \text{hence } \phi = \frac{3\pi}{4} \text{ and;}$$

$$\sin \theta = \frac{1}{2 \cdot \sin \frac{3\pi}{4}} = \frac{1}{2 \cdot \frac{\sqrt{2}\pi}{2}} = \frac{1}{\sqrt{2}}, \quad \text{hence } \theta = \frac{\pi}{4} \text{ or } \frac{3\pi}{4}.$$

But the y-coordinate of $(1, 1, -\sqrt{2})$ has positive sign, so $\theta = \frac{\pi}{4}$. Therefore the desired spherical coordinate is $(2, \frac{\pi}{4}, \frac{3\pi}{4})$.

(b) **(8 points)** In Cartesian coordinates, write down the equation of the surface given by the equation $r = 2\cos\theta$ in cylindrical coordinates and describe the surface in words or in a picture.

Solution: Recall the general formulae for $(x, y, z) \leftrightarrow (r, \theta, z)$

$$x = r\cos\theta; y = r\sin\theta; z = z.$$

Look at

$$r^2 = r \cdot r = r \cdot 2\cos\theta = 2 \cdot r\cos\theta.$$

Note that $r^2 = x^2 + y^2$ and $x = r\cos\theta$. So we have

$$x^2 + y^2 = 2x \text{ or } (x-1)^2 + y^2 = 1.$$

This is the cylinder that is vertical along the z-axis and that has radius 1 centered at $(1, 0, z_0)$ on the plane defined by $z = z_0$, where z_0 is an arbitrary real number.

6. **(17 points)** Let C be the curve given by $\boldsymbol{r}(t) = \langle 2t, \ln t, t^2\rangle$, where ln stands for the natural logarithm.

(a) **(9 points)** Find the arc length of the curve C for $1 \le t \le 4$.

Solution: Note that $\boldsymbol{r}(t)$ traces points only once when t runs over the interval $[1, 4]$ and that $\boldsymbol{r}$ has its derivative $\boldsymbol{r}'(t) = \langle 2, \frac{1}{t}, 2t\rangle$. Now we have the desired arc length

$$\begin{aligned}
\int_1^4 \sqrt{|\boldsymbol{r}'(t)|}\, dt &= \int_1^4 \sqrt{2^2 + \left(\frac{1}{t}\right)^2 + (2t)^2}\, dt \\
&= \int_1^4 \sqrt{4 + \frac{1}{t^2} + 4t^2}\, dt \\
&= \int_1^4 \sqrt{\left(\frac{1}{t} + 2t\right)^2}\, dt \\
&= \int_1^4 \left(\frac{1}{t} + 2t\right) dt \\
&= [\ln t + t^2]_1^4 \\
&= \{\ln 4 + 4^2\} - \{\ln(1) + 1^2\} \\
&= \ln 4 + 15.
\end{aligned}$$

(b) **(8 points)** Find the curvature of the curve C at $t = 1$.

Solution: The curvature is given by

$$\kappa = \frac{|\boldsymbol{r}'(1) \times \boldsymbol{r}''(1)|}{|\boldsymbol{r}'(1)|^3}.$$

Now recall the formula for $\boldsymbol{r}'(t)$ in (a) above and look at

$$\boldsymbol{r}'(1) = \langle 2, 1, 2\rangle \text{ and } |\boldsymbol{r}'(1)| = \sqrt{2^2 + 1^2 + 2^2} = 3.$$

Since $\boldsymbol{r}''(t) = \langle 0, -\frac{1}{t^2}, 2\rangle$, we have

$$\boldsymbol{r}''(1) = \langle 0, -1, 2\rangle.$$

Observe

$$\boldsymbol{r}'(1) \times \boldsymbol{r}''(1) = \langle 2, 1, 2\rangle \times \langle 0, -1, 2\rangle = \langle 4, -4, -2\rangle.$$

(You need to show detailed work for this in the exam.) Therefore we have

$$\kappa = \frac{\sqrt{4^2 + (-4)^2 + (-2)^2}}{3^3} = \frac{6}{27} = \frac{2}{9}.$$