MATH 2400: Calculus 3, Fall 2014
Midterm 3

November 12, 2014

NAME AND SIGNATURE: Solutions

"On my honor, as a University of Colorado at Boulder student, I have neither given nor received unauthorized assistance on this work."

Circle your section.

001 G. Rea .... (9am)

002 J. Migler .... (10am)

003 I. Mishev .... (11am)

004 M. Roy .... (12pm)

005 T. Davison .... (1pm)

006 C. Farsi .... (2pm)

007 D. Monk .... (3pm)

008 K. Havasi .... (9am)

009 J. Nishikawa .... (11am)

You must show all of your work. Please write legibly and box your answers. The use of calculators, books, notes, etc. is not permitted on this exam. Please provide exact answers when possible. For example, if the answer is π, write the symbol "π" and not the decimal 3.14159. . . .

Question	Points	Score
1	15	
2	20	
3	15	
4	20	
5	15	
6	15	
Total:	100	

1. (15 points)

(a) Let T be the triangle in the xy-plane with vertices $(0,0)$, $(1,2)$, and $(2,2)$. Find the area of the surface $z = 7 + \sqrt{8}x + y^2$ above T.

Solution: The partial derivatives are

$$\frac{\partial z}{\partial x} = \sqrt{8}, \quad \frac{\partial z}{\partial y} = 2y$$

The triangle T is given by the inequalities

$$\begin{aligned} 0 \le y \le 2 \\ \frac{1}{2}y \le x \le y \end{aligned}$$

The surface area is given by

$$\begin{aligned} \iint_T \left[1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2\right]^{1/2} dA &= \int_0^2 \int_{\frac{1}{2}y}^{y} \left[9 + 4y^2\right]^{1/2} \, dx\, dy \\ &= \frac{49}{12} \end{aligned}$$

(b) Find the area of the surface with parametrization

$$\vec{r}(u,v) = \left\langle \frac{1}{\sqrt{2}}u^2, uv, \frac{1}{\sqrt{2}}v^2 \right\rangle, \quad 0 \le u \le 1, \quad 0 \le v \le 2.$$

Solution: We calculate

$$\begin{aligned} \vec{r}_u = \langle \sqrt{2}u, v, 0\rangle, &\quad \vec{r}_v = \langle \sqrt{2}v, 0, u\rangle \\ \vec{r}_u \times \vec{r}_v &= \langle \sqrt{2}v^2, -2uv, \sqrt{2}u^2\rangle \\ |\vec{r}_u \times \vec{r}_v|^2 &= 2v^4 + 4u^2v^2 + 2v^4 \\ |\vec{r}_u \times \vec{r}_v| &= \sqrt{2}(u^2 + v^2) \end{aligned}$$

The surface area is given by

$$\begin{aligned} \iint |\vec{r}_u \times \vec{r}_v| \, du\, dv &= \int_0^2 \int_0^1 \sqrt{2}(u^2+v^2)\, du\, dv \\ &= \frac{10\sqrt{2}}{3} \end{aligned}$$

2. (20 points)

   (a) Let S be the solid in the first octant of 3-space bounded by the surfaces $z = \sqrt{x^2+y^2}$ and $x^2+y^2+z^2 = 1$ and the coordinate planes xz and yz. Find and evaluate a triple integral in spherical coordinates that gives the volume of S.

   Solution: The surface $z = \sqrt{x^2+y^2}$ is a cone that makes an angle of $\pi/4$ with the positive z-axis. The first octant is given by $0 \le \theta \le \pi/2$, $0 \le \phi \le \pi/2$. Hence the volume is given by

$$\begin{aligned} \iiint_S dV &= \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{4}} \int_0^1 \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta \\ &= \frac{(2-\sqrt{2})\pi}{12} \end{aligned}$$

   (b) Let R be the region in the first quadrant of the xy-plane bounded by the curves

$$y = e^x, \ y = e^{2x}, \ x = 1.$$

   Evaluate the integral

$$\iint\limits_R \frac{(\ln y)^2}{x^2 y} \, dA$$

   by a change of variables given by

$$x = \frac{v}{u}, \quad y = e^v.$$

   Solution: Under the given change of variables, the integrand becomes

$$\frac{v^2}{\frac{v^2}{u^2} e^v} = u^2 e^{-v}$$

   The partial derivatives are

$$\frac{\partial x}{\partial u} = -\frac{v}{u^2}, \quad \frac{\partial x}{\partial v} = \frac{1}{u}, \quad \frac{\partial y}{\partial u} = 0, \quad \frac{\partial y}{\partial v} = e^v$$

   so the Jacobian determinant is

$$\frac{\partial(x,y)}{\partial(u,v)} = -\frac{ve^v}{u^2}$$

   Finally, the new bounds of integration are

$$\begin{aligned} 1 \le u \le 2 \\ 0 \le v \le u \end{aligned}$$

   and we calculate

$$\begin{aligned} \iint_R \frac{(\ln y)^2}{x^2 y} \, dA &= \int_1^2 \int_0^u u^2 e^{-v} \cdot \frac{ve^v}{u^2} \, dv \, du \\ &= \int_1^2 \int_0^u v \, dv \, du \\ &= \frac{7}{6} \end{aligned}$$

3. (15 points) Let B denote the disc of radius $R > 0$ around the origin and suppose that it has density $\rho(x, y) = \frac{1}{x^2+y^2} e^{-x^2-y^2}$.

   (a) Find the polar moment of inertia around the origin.

   Solution: In polar coordinates, the given density $\rho = r^{-2}e^{-r^2}$. The polar moment of inertia is

$$\begin{aligned} \iint_B r^2 \rho \, dA &= \int_0^{2\pi} \int_0^R e^{-r^2} r \, dr \, d\theta \\ &= (1 - e^{-R^2})\pi \end{aligned}$$

   (b) What happens to this polar moment of inertia as R goes to infinity?

   Solution: As $R \to \infty$, $e^{-R^2} \to 0$, so the polar moment of inertia tends to π.

4. (20 points) (a) Interpret the following iterated integral as the volume of a solid from geometry and use a formula from geometry to evaluate it.

$$\int_0^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{-\sqrt{4-x^2-y^2}}^{\sqrt{4-x^2-y^2}} dz\, dy\, dx$$

Solution: The region of integration is half of a solid ball of radius 2, so the integral is

$$\frac{1}{2} \cdot \frac{4}{3}\pi(2)^3 = \frac{16\pi}{3}$$

(b) Find the volume of the solid bounded by the parabolic cylinders $y = x^2$ and $x = y^2$ and the planes $z = 0$ and $z = 2x + y$.

Solution: The region R described in the xy-plane is given by the inequalities

$$\begin{gathered} 0 \leq x \leq 1 \\ x^2 \leq y \leq \sqrt{x} \end{gathered}$$

The volume is given by

$$\begin{aligned} \iint_R (2x+y)\, dA &= \int_0^1 \int_{x^2}^{\sqrt{x}} (2x+y)\, dy\, dx \\ &= \frac{9}{20}. \end{aligned}$$

(c) Evaluate the following iterated integral by first changing the order of integration.

$$\int_0^2 \int_0^3 \int_0^{1-z/2} e^{2x-x^2}\,dx\,dy\,dz$$

Solution:

$$\begin{aligned}\int_0^2 \int_0^3 \int_0^{1-z/2} e^{2x-x^2}\,dx\,dy\,dz &= \int_0^1 \int_0^3 \int_0^{2-2x} e^{2x-x^2}\,dz\,dy\,dx \\ &= \int_0^1 3(2-2x)e^{2x-x^2}\,dx\end{aligned}$$

Letting $u = 2x - x^2$, we find

$$\begin{aligned}\int_0^1 3(2-2x)e^{2x-x^2}\,dx &= \int_{u=0}^{u=1} 3e^u\,du \\ &= 3(e-1)\end{aligned}$$

5. (15 points) (a) Evaluate the following double integral:

$$\int_0^{\frac{\pi}{2}} \int_x^{\frac{\pi}{2}} \frac{\sin^2(y)}{y} dy\, dx$$

Solution: By changing the order of integration, we find

$$\begin{aligned}\int_0^{\frac{\pi}{2}} \int_x^{\frac{\pi}{2}} \frac{\sin^2(y)}{y} dy\, dx &= \int_0^{\frac{\pi}{2}} \int_0^{y} \frac{\sin^2(y)}{y} dx\, dy \\ &= \int_0^{\frac{\pi}{2}} \sin^2 y\, dy \\ &= \frac{\pi}{4}\end{aligned}$$

(b) Evaluate the double integral of the function $f(x, y) = \frac{2y}{x+1}$ over the region in the xy-plane bounded by $y = \sqrt{x}$, $x = 0$, $x = 1$, and the x-axis.

Solution: The desired integral is

$$\begin{aligned}\int_0^1 \int_0^{\sqrt{x}} \frac{2y}{x+1}\, dy\, dx &= \int_0^1 \left[\frac{y^2}{x+1} \Big|_{y=0}^{\sqrt{x}} \right] dx \\ &= \int_0^1 \frac{x}{x+1}\, dx\end{aligned}$$

Letting $u = x + 1$, we find

$$\begin{aligned}\int_0^1 \frac{x}{x+1}\, dx &= \int_{u=1}^{2} \frac{u-1}{u}\, du \\ &= 1 - \ln 2\end{aligned}$$

6. (15 points) (a) Use polar coordinates to evaluate the following integral in the region between $x^2+y^2=1$ and $x^2+y^2=2$

$$\iint_R \frac{1}{x^2+y^2}\,dA$$

Solution: In polar coordinates, $x^2+y^2=r^2$, so

$$\begin{aligned}\iint_R \frac{1}{x^2+y^2}\,dA &= \int_0^{2\pi}\int_1^{\sqrt{2}} \frac{1}{r^2}\,r\,dr\,d\theta \\ &= \pi \ln 2.\end{aligned}$$

(b) Find the mass of the planar lamina in region R with density $\rho(x,y)=\sqrt{x^2+y^2}$ where R is the region created by the trace of the $x^2+y^2+(z-4)^2=20$ in the xy plane.

Solution: R is the region inside the circle $x^2+y^2=4$. In polar coordinates, the given density $\rho=r$, so the mass is

$$\begin{aligned}\iint_R \rho\,dA &= \int_0^{2\pi}\int_0^{2} r\cdot r\,dr\,d\theta \\ &= \frac{16\pi}{3}.\end{aligned}$$