Math 2400: Midterm 2 Review

1. Let $p_0 = (1, 0, 2)$. In 3-dimensional space, identify the range of each of the following as either a line, a plane, or neither:

   (a) $\vec{r_1}(t) = p_0 + t < -2, 0, 1 >$
   Answer: This is a line.

   (b) $\vec{r_2}(t) = p_0 + t^2 < -2, 0, 1 >$
   Answer: This is *neither* a line nor a plane. As t ranges over all real numbers t^2 ranges over $[0, \infty)$ so that $\vec{r_2}(t)$ traces out the ray starting at p_0 and going off to infinity in the direction $< -2, 0, 1 >$.

   (c) $\vec{r_3}(t) = p_0 + t^3 < -2, 0, 1 >$
   Answer: This is a line. As t ranges over all real numbers t^3 will also range over all real numbers so that the range of $\vec{r_3}(t)$ will trace out a line.

   (d) $\vec{r_4}(t) =< 1 - 2t + t^2, t, -2t >$
   Answer: This is *neither* a line nor a plane. Because the x-coordinate has the equation of a parabola (i.e. $1 - 2t + t^2$), the x-coordinate will trace out a parabola while the y and z-coordinates will trace out lines. If the t^2 was not there then it would trace out a line.

   (e) $\vec{r_5}(s, t) = p_0 + s < 1, -1, 2 > +t < -2, 0, 1 >$
   Answer: This will trace out a plane.

   (f) $\vec{r_6}(s, t) = p_0 + s < -2, 0, 1 > +t < -2, 0, 1 >$
   Answer: This will trace out a line. Note that $\vec{r_6}(s, t) = p_0 + s < -2, 0, 1 > +t < -2, 0, 1 >= p_0 + (s + t) < -2, 0, 1 >$ always lies on the line $\vec{r_1}(t)$ so it can't be a plane. As s and t both vary (independently) over all real numbers, $s + t$ will vary over all real numbers so the range of $\vec{r_6}(t)$ is a line.

   (g) $\vec{r_7}(s, t) = p_0 + s < 0, 0, 0 > +t < -2, 0, 1 >$
   Answer: This will trace out a line. Note that $\vec{r_7}(s, t) = p_0 + s < 0, 0, 0 > +t < -2, 0, 1 >= p_0 + t < -2, 0, 1 >$ so that s doesn't affect the range.

   (h) $\vec{r_8}(s, t) =< 1 - 2s^2 + t, t, 2 + s + 2t^2 >$
   Answer: This is *neither* a line nor a plane.

2. Let $x(t) = 1 + t$, $y(t) = -t$, and $z(t) = 2 + 2t$. Write this symmetric equations of this line and write the vector equation of this line.
   Answer: To write the symmetric equations of this line, we first solve each of the above equations for t: $t = x - 1$, $t = -y$, and $t = \frac{z-2}{2}$. Then the symmetric equations are $x - 1 = -y = \frac{z-2}{2}$. The vector equation is $\vec{r}(t) =< x(t), y(t), z(t) >=< 1 + t, -t, 2 + 2t >=< 1, 0, 2 > + < t, -t, 2t >=< 1, 0, 2 > +t < 1, -1, 2 >$.

3. Find the domain and derivative/partial derivatives of each of the following:

   (a) $\vec{r_1}(t) = \langle \sqrt{9 - t^2}, \ln(t - 1), e^{t^2} \rangle$
   Answer: We need $9 - t^2 \geq 0$ for $\sqrt{9 - t^2}$ to be defined, which implies that $9 \geq t^2$, which happens exactly when $|t| < 3$ i.e. when $-3 \leq t \leq 3$. For $\ln(t - 1)$ to be defined, we need to have $t - 1 > 0$ so that $t > 1$. Since e^{t^2} is always defined so it places no restrictions on the domain of $\vec{r_1}(t)$. So the domain is $(1, 3]$.

Note that e^{t^2} means $e^{(t^2)}$ and *not* $(e^t)^2$ (since $(e^t)^2 = e^te^t = e^{2t}$ so we would write e^{2t} instead of $(e^t)^2$). So the derivative is $\vec{r_1}'(t) = \left\langle \frac{d}{dt}\left(\sqrt{9-t^2}\right), \frac{d}{dt}\ln(t-1), \frac{d}{dt}e^{t^2}\right\rangle = \left\langle \frac{1}{2}(9-t^2)^{-1/2}(-2t), \frac{1}{t-1}, 2te^{t^2}\right\rangle = \left\langle \frac{-t}{\sqrt{9-t^2}}, \frac{1}{t-1}, 2te^{t^2}\right\rangle$.

(b) $\vec{r_2}(t) = \langle \frac{1}{\pi^2-4t^2}, \tan t, \arcsin t\rangle$
Answer: $\frac{1}{\pi^2-4t^2}$ is *un*defined exactly when $0 = \pi^2 - 4t^2 = \pi^2 - (2t)^2$ which happens exactly when $2t = \pm\pi$ so that $\frac{1}{\pi^2-4t^2}$ is defined exactly when $t \neq \pm\frac{\pi}{2}$. $\tan t$ is defined exactly when $t \neq \frac{\pi}{2} + n\pi$ for any integer n. Note that since $n = 0$ and $n = -1$ give us $t \neq \frac{\pi}{2}$ and $t \neq -\frac{\pi}{2}$, the restriction on the domain that $\tan t$ gives subsumes the restriction that $\frac{1}{\pi^2-4t^2}$ gave us. For $\arcsin t$ to be defined, we need to have $-1 \leq t \leq 1$. Since all $\frac{\pi}{2} + n\pi$ (for integers n) are not in this interval we have that the requirement $-1 \leq t \leq 1$ is stronger than the requirement on the domain that is given by $\arctan t$. Thus the domain of $\vec{r_2}(t)$ is $[-1, 1]$.
The derivative is $\vec{r_2}'(t) = \left\langle \frac{d}{dt}\left(\frac{1}{\pi^2-4t^2}\right), \frac{d}{dt}\tan t, \frac{d}{dt}\arcsin t\right\rangle = \left\langle \frac{8t}{(\pi^2-4t^2)^2}, \sec^2 t, \frac{1}{\sqrt{1-t^2}}\right\rangle$.

(c) $\vec{r_3}(s,t) = \langle \frac{1}{\pi^2-4s^2}, \tan s, \arcsin t\rangle$
Answer: By using the reason from $\vec{r_2}(t)$ we see that we $\vec{r_3}(s,t)$ is defined exactly when $s \neq \frac{\pi}{2} + n\pi$ for any integer n and $-1 \leq t \leq 1$. The partial derivatives with respect to s and t are
$\frac{\partial \vec{r_3}}{\partial s}(s,t) = \left\langle \frac{\partial}{\partial s}\left(\frac{1}{\pi^2-4s^2}\right), \frac{\partial}{\partial s}\tan s, \frac{\partial}{\partial s}\arcsin t\right\rangle = \left\langle \frac{8s}{(\pi^2-4s^2)^2}, \sec^2 s, 0\right\rangle$ and
$\frac{\partial \vec{r_3}}{\partial t}(s,t) = \left\langle \frac{\partial}{\partial t}\left(\frac{1}{\pi^2-4s^2}\right), \frac{\partial}{\partial t}\tan s, \frac{\partial}{\partial t}\arcsin t\right\rangle = \left\langle 0, 0, \frac{1}{\sqrt{1-t^2}}\right\rangle$.

(d) $\vec{r_4}(t) = \langle \sin t, \cos t, e^t\rangle$
Answer: If we plug any real number into $\vec{r_4}(t)$ then we will be able to compute resulting vector so this means exactly that the domain consists of all real numbers. The derivative with respect to t is $\vec{r_4}'(t) = \langle \frac{d}{dt}\sin t, \frac{d}{dt}\cos t, \frac{d}{dt}e^t\rangle = \langle \cos t, -\sin t, e^t\rangle$.

4. Let $r(t) = \langle e^t\cos t, e^t\sin t, e^t\rangle$. Compute $T(t)$, $N(t)$, $B(t)$, the curvature $\kappa(t)$, and the length of $r(t)$ from a to b.
Answer: $r'(t) = \langle e^t\cos t - e^t\sin t, e^t\sin t + e^t\cos t, e^t\rangle = e^t\langle \cos t - \sin t, \sin t + \cos t, 1\rangle$.
Note that $(\cos t - \sin t)^2 + (\cos t + \sin t)^2 = (\cos^2 - 2\sin t\cos t + \sin^2 t) + (\cos^2 + 2\sin t\cos t + \sin^2 t) = 2$ so $|r'(t)| = e^t|\langle \cos t - \sin t, \sin t + \cos t, 1\rangle| = e^t\sqrt{(\cos t - \sin t)^2 + (\cos t + \sin t)^2 + 1^2} = e^t\sqrt{2+1} = \sqrt{3}e^t$.
So $T(t) = \frac{r'(t)}{|r'(t)|} = \frac{e^t\langle \cos t - \sin t, \sin t + \cos t, 1\rangle}{\sqrt{3}e^t} = \frac{1}{\sqrt{3}}\langle \cos t - \sin t, \sin t + \cos t, 1\rangle$.
Now, $T'(t) = \frac{1}{\sqrt{3}}\langle -\sin t - \cos t, \cos t - \sin t, 0\rangle$ and so $|T'(t)| = \frac{1}{\sqrt{3}}\sqrt{(-\sin t - \cos t)^2 + (\cos t - \sin t)^2 + 0^2} = \frac{1}{\sqrt{3}}\sqrt{(\sin t + \cos t)^2 + (\cos t - \sin t)^2} = \frac{1}{\sqrt{3}}\sqrt{2} = \sqrt{\frac{2}{3}}$.
So $N(t) = \frac{T'(t)}{|T'(t)|} = \frac{\frac{1}{\sqrt{3}}\langle -\sin t - \cos t, \cos t - \sin t, 0\rangle}{\sqrt{\frac{2}{3}}} = \frac{1}{\sqrt{2}}\langle -\sin t - \cos t, \cos t - \sin t, 0\rangle$ and
$B(t) = T(t) \times N(t) = \vec{i}[-\frac{1}{\sqrt{6}}(\cos t - \sin t)] - \vec{j}[-\frac{1}{\sqrt{6}}(-\cos t - \sin t)] + \vec{k}\frac{1}{\sqrt{6}}[(\cos t - \sin t)^2 + (\cos t + \sin t)^2] = \frac{1}{\sqrt{6}}\langle \sin t - \cos t, -\sin t - \cos t, \sqrt{2}\rangle$.
Since we've already computed $|T'(t)|$ and $|r'(t)|$ to compute the curvature we will use $\kappa(t) = \frac{|T'(t)|}{|r'(t)|} = \frac{\sqrt{\frac{2}{3}}}{\sqrt{3}e^t} = \sqrt{2}e^{-t}$.
The length of $r(t)$ from a to b is $\int_a^b |r'(t)|dt = \int_a^b \sqrt{3}e^t dt = \sqrt{3}e^t|_a^b = \sqrt{3}[e^b - e^a]$.

5. True or False (and justify your answer): Let $f(t)$ be a real-valued function and let $\vec{u}(t)$, $\vec{v}(t)$, and $\vec{r}(t)$ be vector valued functions.

   (a) $\frac{d}{dt}\left(f(t)\vec{u}(t)\right) = f'(t)\vec{u}(t) + f(t)\vec{u}'(t)$
   True.

   (b) $\frac{d}{dt}\left(\vec{u}(t)\cdot\vec{v}(t)\right) = \vec{u}'(t)\cdot\vec{v}'(t)$
   False: The correct formula is $\frac{d}{dt}\left(\vec{u}(t)\cdot\vec{v}(t)\right) = \vec{u}'(t)\cdot\vec{v}(t) + \vec{u}(t)\cdot\vec{v}'(t)$.

   (c) $\frac{d}{dt}\left(\vec{u}(t)\times\vec{v}(t)\right) = \vec{u}'(t)\times\vec{v}'(t)$
   False: The correct formula is $\frac{d}{dt}\left(\vec{u}(t)\times\vec{v}(t)\right) = \vec{u}'(t)\times\vec{v}(t) + \vec{u}(t)\times\vec{v}'(t)$.

   (d) If $\vec{r}(t) \neq 0$ then $\frac{d}{dt}|r(t)| = \frac{d}{dt}\sqrt{r(t)\cdot r(t)} = \frac{1}{|r(t)|}r(t)\cdot r'(t)$.
   True. Recall that $|r(t)|^2 = r(t)\cdot r(t)$ so that the first equality is true. Since $r(t) \neq 0$, $r(t)$ is a point at which the magnitude function (i.e. the function that assigns to a vector $\vec{v} = \langle x, y, z\rangle$ its length $|\vec{v}| = \sqrt{x^2+y^2+z^2}$) is differentiable so that by the chain rule $\frac{d}{dt}\left(|r(t)|^2\right) = 2|r(t)|\frac{d}{dt}\left(|r(t)|\right)$. But also, $\frac{d}{dt}\left(|r(t)|^2\right) = \frac{d}{dt}\left(r(t)\cdot r(t)\right) = r'(t)\cdot r(t) + r(t)\cdot r'(t) = 2\vec{r}(t)\cdot\vec{r}'(t)$. So we have $2|r(t)|\frac{d}{dt}\left(|r(t)|\right) = \frac{d}{dt}\left(|r(t)|^2\right) = 2\vec{r}(t)\cdot\vec{r}'(t)$. Dividing by $2|r(t)|$ gives us the second equality.

   (e) If $\vec{r}(t) = \langle t, f(t)\rangle$ then the length of $f(t)$ from $t = a$ to $t = b$ is the same as the length of $r(t)$ from a to b.
   True. Recall that the length of $f(t)$ from a to b is $\int_a^b \sqrt{1+f'(t)}dt$ while the length of $r(t)$ from a to b is $\int_a^b |r'(t)|dt = \int_a^b \left|\left\langle \frac{d}{dt}t, \frac{d}{dt}f(t)\right\rangle\right| dt = \int_a^b |\langle 1, f'(t)\rangle|\, dt$.

   (f) Let $T(t)$ be the unit tangent vector of $r(t)$ and let $N(t)$ be the unit normal vector of $r(t)$. Then the unit tangent vector of $T(t)$ is $N(t)$.
   True. The unit tangent vector of $T(t)$ is $\frac{T'(t)}{|T'(t)|}$, which is the same as the the unit normal vector of $r(t)$.

   (g) Fix a point p_0 in space that $r(t)$ goes through. Then the curvature of the curve $r(t)$ at p_0 depends on the paramterization of $r(t)$.
   False. One reason that curvature, $\kappa = \left|\frac{dT}{ds}\right|$, uses the derivative with respect to the arclength of the curve (as opposed to using the derivative with respect to the parameter t) is so that it will not be dependent on how the curve is parameterized.

   (h) The curvature of $r(t)$ is $\kappa(t) = \frac{|T'(t)|}{|r'(t)|} = \frac{|r'(t)\times r''(t)|}{|r'(t)|^3}$.
   True. Note that although this equation makes it seem as though the curvature depends on the paramterization of the curve, it is in fact independent of the paramterization of $r(t)$.

   (i) It is possible for $T(t)$ to point in the opposite direction of $r'(t)$.
   False. Since $T(t) = \frac{r'(t)}{|r'(t)|}$ and $|r'(t)|$ is never negative, $T(t)$ and $r'(t)$ will always point in the same direction.

   (j) $T(t)$ and $N(t)$ are perpendicular.
   True.

   (k) If $s(t)$ is the arclength function of $\vec{r}(t)$ (starting at some arbitrary $t = a$) then $\frac{ds}{dt} = |r'(t)|$.
   True. $s(t) = \int_a^t |r'(t)|dt$ so by the fundamental theorem of calculus, $\frac{ds}{dt} = |r'(t)|$.

6. Let $\vec{r}(t)$ be a vector valued function. What is wrong with the following reasoning?
   The length of $r(t)$ from $t = a$ to $t = b$ is $L = \int_a^b |r'(t)|dt = |\int_a^b r'(t)dt| = |r(t)|_a^b| = |r(b) - r(a)|$.
   Answer: The mistake is that the equality $\int_a^b |r'(t)|dt = |\int_a^b r'(t)dt|$ is almost never true.

7. Let $\vec{r}(t)$, $\vec{v}(t)$, and $\vec{a}(t)$ denote the position, velocity, and acceleration of a particle at time t. Suppose that $\vec{r}(0) = \langle 1, \ln 4, 1\rangle$ and $\vec{v}(0) = \langle 1, 1, 0\rangle$. For each of the below, find all three of $\vec{r}(t)$, $\vec{v}(t)$, and $\vec{a}(t)$.

   (a) $\vec{r}(t) = \langle e^{\sin t}, \ln((2+t)^2), \cos t\rangle$.
   Answer: Note that $\ln((2+t)^2) = 2\ln(2+t)$ so that $\vec{v}(t) = \left\langle e^{\sin t}\cos t, \frac{2}{2+t}, -\sin t\right\rangle$ and so $\vec{a}(t) = \vec{v}'(t) = \left\langle e^{\sin t}(\cos^2 t - \sin t), \frac{-2}{(2+t)^2}, -\cos t\right\rangle$.

   (b) $\vec{v}(t) = \langle \frac{1}{1+t^2}, \frac{2e^t}{1+e^t}, \sin t\cos t\rangle$.
   Answer: Using the fact that $\sin(2t) = 2\sin t\cos t$, we have that $\vec{r}(t) = \int \vec{v}(t)dt = \int\langle \frac{1}{1+t^2}, \frac{2e^t}{1+e^t}, \frac{1}{2}\sin(2t)\rangle dt = \langle \arctan t, 2\ln|1+e^t|, \frac{-1}{4}\cos(2t)\rangle + \vec{C}$, where $\vec{C} = \langle C_1, C_2, C_3\rangle$. Since $\langle 1, \ln 4, 1\rangle = \vec{r}(0) = \langle \arctan 0, 2\ln|1+e^0|, \frac{-1}{4}\cos(20)\rangle + \vec{C} = \langle 0, 2\ln 2, \frac{-1}{4}\rangle + \vec{C}$ we have $\vec{C} = \langle 1, \ln 4, 1\rangle - \langle 0, 2\ln 2, \frac{-1}{4}\rangle = \langle 1, 0, \frac{5}{4}\rangle$.
   Hence, $\vec{r}(t) = \langle \arctan t + 1, 2\ln|1+e^t|, \frac{-1}{4}\cos(2t) + \frac{5}{4}\rangle$.
   Now, $\vec{a}(t) = \vec{v}'(t) = \left\langle \frac{d}{dt}\frac{1}{1+t^2}, \frac{d}{dt}\frac{2e^t}{1+e^t}, \frac{d}{dt}\frac{1}{2}\sin(2t)\right\rangle = \left\langle \frac{-2t}{(1+t^2)^2}, \frac{2e^t(1+e^t)-e^t(2e^t)}{(1+e^t)^2}, \cos(2t)\right\rangle = \left\langle \frac{-2t}{(1+t^2)^2}, \frac{2e^t}{(1+e^t)^2}, \cos(2t)\right\rangle$.

   (c) $\vec{a}(t) = \left\langle e^t, -\frac{1}{(1+t)^2}, -\cos t\right\rangle$.
   Answer: We have that $\vec{v}(t) = \int \vec{a}(t)dt = \int\left\langle e^t, -\frac{1}{(1+t)^2}, -\cos t\right\rangle dt = \left\langle e^t, \frac{1}{1+t}, -\sin t\right\rangle + \vec{D}$, where $\vec{D} = \langle D_1, D_2, D_3\rangle$. Since $\vec{v}(0) = \langle 1, 1, 0\rangle$ we get $\vec{D} = \langle 0, 0, 0\rangle$.
   Looking ahead, noticing that $\int \frac{1}{1+t}dt = \ln|1+t|$ and considering that we want $\vec{r}(0) = \langle 1, \ln 4, 1\rangle$ we integrate $\frac{1}{1+t} = \frac{4}{4+4t}$ to get that the position is $\vec{r}(t) = \int \vec{v}(t)dt = \int\left\langle e^t, \frac{4}{4+4t}, -\sin t\right\rangle dt = \left\langle e^t, \ln|4+4t|, \cos t\right\rangle + \vec{C}$ where we conclude that $\vec{C} = \langle 0, 0, 0\rangle$ (explain why our answer would not have been different if we had integrated $\frac{1}{1+t}$ instead of $\frac{4}{4+4t}$. How would your work have changed?).

8. Parameterize the following surfaces:

   (a) The sphere of radius R centered at the origin.
   Answer: We can parameterize the sphere using spherical coordinates: $x(\phi,\theta) = R\sin\phi\cos\theta$, $y(\phi,\theta) = R\sin\phi\sin\theta$, and $z(\phi,\theta) = R\cos\phi$ with domain $[0,\pi]\times[0,2\pi]$.

   (b) The sphere of radius R centered at $\vec{p_0} = \langle a, b, c\rangle$.
   Answer: $x(\phi,\theta) = a + R\sin\phi\cos\theta$, $y(\phi,\theta) = b + R\sin\phi\sin\theta$, and $z(\phi,\theta) = c + R\cos\phi$ with domain $[0,\pi]\times[0,2\pi]$.

   (c) The plane that contains the lines $\vec{p_0} + t\vec{v}$ and $\vec{p_0} + t\vec{w}$, where $\vec{p_0} = \langle a, b, c\rangle$, $\vec{v} = \langle v_1, v_2, v_3\rangle$, $\vec{w} = \langle w_1, w_2, w_3\rangle$, and where we assume that $\vec{v}$ and $\vec{w}$ are not parallel.
   Answer: We will use the usual parameterization: $\vec{r}(s,t) = p_0 + s\vec{v} + t\vec{w}$ or equivalently, $x(s,t) = a + sv_1 + tw_1$, $y(s,t) = b + sv_2 + tw_2$, and $z(s,t) = c + sv_3 + tw_3$ with domain $(-\infty,\infty)\times(-\infty,\infty)$.

   (d) Suppose that P is a plane that goes through the origin and has normal vector $\vec{n} = \langle n_1, n_2, n_3\rangle$. Suppose we know that $n_3 \neq 0$. Find equations that parameterize P.
   Answer: Since P goes through the origin, if we can find two non-zero and non-parallel vector $\vec{v}$ and $\vec{w}$ that are contained in P then by using the parameterization of the plane from above, we can parameterize P. Since we need *any* vectors $\vec{v}$ and $\vec{w}$ that are non-zero

and non-parallel we can (after some thought) hope that we can pick $\vec{v}$ and $\vec{w}$ so that they have the form $\vec{v} = \langle 1, 0, c \rangle$ and $\vec{w} = \langle 0, 1, d \rangle$ for some unknown constant c and d that we now wish to find. Since P goes through the origin, for $\vec{v}$ to be contained in P we need $0 = \vec{n} \cdot \vec{v} = \langle n_1, n_2, n_3 \rangle \cdot \langle 1, 0, c \rangle = n_1 + cn_3$, which forces $c = -\frac{n_1}{n_3}$ (this is defined since $n_3 \neq 0$). Similarly, $d = -\frac{n_2}{n_3}$. And so we can use the parameterization $\vec{r}(s,t) = p_0 + s\vec{v} + t\vec{w}$ as described above.

(e) Suppose that we have given a real-valued function $f(y)$ with domain D, where D is a set of real-numbers. Find a parameterization for the surface of revolution obtained by revolving $f(y)$ around the y-axis.
Answer: One possible parameterization is $x(y,\theta) = f(y)\cos\theta$, $y = y$, and $z(y,\theta) = f(y)\sin\theta$ with domain $D \times [0, 2\pi)$.

(f) What surface does $\vec{r}(r,\theta) = \langle r\cos\theta, r, r\sin\theta \rangle$ parameterize?
Answer: A right circular cone.

(g) Suppose that $f(y,z)$ is a real-valued function. Find a parameterization for the graph of $f(y,z)$. Apply it to the specific case of $f(y,z) = y^2 + 2z^2$.
Answer: $x = f(y,z)$, $y = y$, and $z = z$ is a paramterization. For the case of $f(y,z) = y^2 + 2z^2$, the paramterization is $x = y^2 + 2z^2$, $y = y$, and $z = z$.

9. Draw the level surfaces of $f(x,y) = \sqrt{(x-2)^2 + (y+1)^2}$ and then describe the graph of this function.what is the domain of this function?
Answer: The level surfaces are of the form $c = f(x,y) = \sqrt{(x-2)^2 + (y+1)^2}$, which is only valid if $c \geq 0$ (since $f(x,y)$ is a function, the square root only outputs the positive root). And this equation becomes $(x-2)^2 + (y+1)^2 = c^2$, which is a circle or radius c centered around the point $(2, -1)$. Since the radius is exactly c, we see that the graph of this function is the upper half of a right circular cone centered at $(2, -1)$. The domain of this function is $(-\infty, \infty) \times (-\infty, \infty)$.

10. Draw a contour map of the function $f(x,y) = ye^{-x}$ showing several level curves.
Answer: The level curves are $c = f(x,y) = ye^{-x}$ which gives $y = ce^x$. Note that these level curves are drawn in the x-y plane (not in 3-D). If $c = 0$ then it is just the line $y = 0$. Otherwise, if $c > 0$ then it looks like a vertical scaling (with scaling factor c) of the graph of e^x and if $c < 0$ then it looks like a vertical scaling (with scaling factor $|c|$) of the graph of e^x followed by a flip across the x-axis. When drawing a contour map, make sure to increase the value of c uniformly by some constant step size (ex: maybe draw the level curves for $c = -4$, $c = -2$, $c = 0$, $c = 2$, $c = 4$, where we happened to have picked a step size of 2). Also include any special case (ex: in this problem it was important to draw the level curve of $c = 0$ since it is a special case).

11. State Clairaut's Theorem.
Answer: See Stewart p. 763. Suppose that f is defined on a disk D that contains the point (a,b). If the functions f_{xy} and f_{yx} both exist and are continuous on D then $f_{xy}(a,b) = f_{yx}(a,b)$.

12. True or False (and justify your answer): Let $f(x,y)$ be a real-valued function.

(a) If $f_x(x,y)$ and $f_y(x,y)$ exist for all x and y, then $f(x,y)$ is continuous.
Answer: False. If $f(x,y) = \frac{xy}{x^2+y^2}$ for $(x,y) \neq (0,0)$ and $f(0,0) = 0$ then $f_x(x,y)$ and $f_y(x,y)$ exist for all x and y but $f(x,y)$ is *not* continuous.

(b) If $f_x(x,y)$ and $f_y(x,y)$ exist for all x and y and both are continuous functions then $f(x,y)$ is continuous.

Answer: True. This is a special case of Stewart p. 773 Theorem 8. Note that for the counter-example above, although the partial derivatives exist everywhere they are not continuous so this theorem does not apply in that situation.

(c) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4}$ exists.

Answer: False. This limit does not exist. Along the line $x = 0$, we have $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4} = \lim_{y\to 0} \frac{0y^2}{0^2+y^4} = 0$ while along $x = y^2$ we have $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4} = \lim_{y\to 0} \frac{y^2y^2}{(y^2)^2+y^4} = \lim_{y\to 0} \frac{1}{2} = \frac{1}{2}$.

(d) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy^4}{x^2+y^4}$ exists.

Answer: True. Note that $\left|\frac{xy^4}{x^2+y^4}\right| = |x|\left|\frac{y^4}{x^2+y^4}\right| \leq |x|1 = |x|$ so that by the squeeze theorem, as $(x, y) \to (0, 0)$, we have $\displaystyle\lim_{(x,y)\to(0,0)} \left|\frac{xy^4}{x^2+y^4} - 0\right| \leq \lim_{(x,y)\to(0,0)} |x| = \lim_{x\to 0} |x| = 0$ so $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy^4}{x^2+y^4} = 0$.

(e) Suppose that f is defined on an open disk D. If all of f's partial derivatives exist and are continuous on D then $f_{xyy} = f_{yyx}$ at every point of D.
Answer: True. By Clairaut's Theorem, $f_{xy} = f_{yx}$ at every point of D so that that by taking this functions partial derivative with respect to y we get $f_{xyy} = f_{yxy}$. By Clairaut's Theorem applied to f_y this time, we have that $f_{yxy} = (f_y)_{xy} = (f_y)_{yx} = f_{yyx}$ at every point of D. Hence, at every point of D we have $f_{xyy} = f_{yxy} = f_{yyx}$.

13. Find the total differential of $z = f(x, y) = x^2 + 3xy - y^2$ and use it to estimate $f(2.05, 2.96)$.
Answer: We have $\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(x^2+3xy-y^2) = \frac{\partial}{\partial x}(x^2)+3y\frac{\partial}{\partial x}(x) - \frac{\partial}{\partial x}(y^2) = 2x+3y+0 = 2x+3y$. Similarly, $\frac{\partial f}{\partial y} = -2y + 3x$. So the total differential is $dz = (2x + 3y)dx + (-2y + 3x)dy$. Since $f(2.05, 2.96)$ is difficult to compute while $f(2, 3)$ is easy to compute and since $(2, 3)$ is close to $(2.05, 2.96)$ we let $(x, y) = (2, 3)$ and $(\Delta x, \Delta y) = (2.05, 2.96) - (2, 3) = (0.05, -0.04)$ we have that $\Delta z \approx (2(2) + 3(3))(0.05) + (-2(3) + 3(2))(-0.04) = 13(0.05) + 0 = 13\frac{5}{100} = \frac{65}{100} = 0.65$ or we could equivalently express this number as the fraction $\Delta z \approx 13\frac{1}{20} = \frac{13}{20}$. Note that no calculator was necessary to compute this approximation of $f(2.05, 2.96)$.

14. Find the tangent plane to $f(x, y) = x^2 + 3xy - y^2$ at $(x, y) = (2, 3)$.
Answer: Using the computations from the problem above, we have that $\frac{\partial f}{\partial x}(2, 3) = 13$ and $\frac{\partial f}{\partial y}(2, 3) = 0$. Since $f(2, 3) = 2^2 + 3(2)(3) - 3^2 = 4 + 18 - 9 = 13$, the equation of the tangent plane is $z = f(2, 3) + f_x(2, 3)(x - 2) + f_y(2, 3)(y - 3) = 13 + 13(x - 2) + 0(y - 3) = 13(x - 1)$.

15. Find an equation for the tangent plane to the surface given by

$$xy + yz^2 + z = 0$$

at the point $(-2, 1, 1)$.
Answer: Let $f(x, y, z) = xy+yz^2+z$. Then the surface described is the level surface $f(x, y, z) = 0$. We calculate that

$$\nabla f = y\vec{i} + (x + z^2)\vec{j} + (2yz + 1)\vec{k},$$

meaning that the vector

$$(\nabla f)(-2,1,1) = \vec{i} - \vec{j} + 3\vec{k}$$

is normal to the tangent plane at $(-2,1,1)$. Using the equation for a plane given a point on the plane and a normal vector to the plane, an equation for the tangent plane is

$$(x+2) - (y-1) + 3(z-1) = 0$$

16. Suppose that $xyz = \cos(x+y+z)$. *Use implicit differentiation* to find $\frac{\partial y}{\partial x}$ and $\frac{\partial y}{\partial z}$.
Answer: Differentiating both sides with respect to x gives us $\frac{\partial}{\partial x}(xyz) = \frac{\partial}{\partial x}(\cos(x+y+z))$, which becomes by the product rule and the chain rule $yz + xz\frac{\partial y}{\partial x} + xy\frac{\partial z}{\partial x} = -\sin(x+y+z)\left(1 + \frac{\partial y}{\partial x} + \frac{\partial z}{\partial x}\right)$. Since we're considering y as a function of the independent variables x and z we have $\frac{\partial z}{\partial x} = 0$ so that $yz + xz\frac{\partial y}{\partial x} = -\sin(x+y+z)\left(1 + \frac{\partial y}{\partial x}\right)$. This becomes $yz + \sin(x+y+z) = -\sin(x+y+z)\frac{\partial y}{\partial x} - xz\frac{\partial y}{\partial x} = -\frac{\partial y}{\partial x}(\sin(x+y+z) + xz)$. Dividing, we get $\frac{\partial y}{\partial x} = \frac{\sin(x+y+z)+yz}{-\sin(x+y+z)-xz}$. Similarly, $yz\frac{\partial x}{\partial z} + xz\frac{\partial y}{\partial z} + xy = \frac{\partial}{\partial z}(xyz) = \frac{\partial}{\partial z}(\cos(x+y+z)) = -\sin(x+y+z)\left(\frac{\partial x}{\partial z} + \frac{\partial y}{\partial z} + 1\right)$ which becomes $xz\frac{\partial y}{\partial z} + xy = -\sin(x+y+z)\left(\frac{\partial y}{\partial z} + 1\right)$. Solving for $\frac{\partial y}{\partial z}$ gives us $\frac{\partial y}{\partial z} = \frac{-\sin(x+y+z)-xy}{\sin(x+y+z)+xz}$.

17. Let $w = f(t)g(t)h(t)$. Use the chain rule applied to $p(x,y,z) = f(x)g(y)h(z)$ to calculate $\frac{dw}{dt}$.
Answer: Note that $w = p(t,t,t)$, which is the same as saying that $w = p(x(t), y(t), z(t))$, where $x(t) = y(t) = z(t) = t$. The chain rule tells us that

$$\begin{aligned}
\frac{dw}{dt} &= \frac{dp}{dt} \\
&= \frac{\partial p}{\partial x}\frac{dx}{dt} + \frac{\partial p}{\partial y}\frac{dy}{dt} + \frac{\partial p}{\partial x}\frac{dz}{dt} \\
&= \frac{\partial p}{\partial x} + \frac{\partial p}{\partial y} + \frac{\partial p}{\partial x} \\
&= f'(x(t))g(y(t))h(z(t)) + f(x(t))g'(y(t))h(z(t)) + f(x(t))g(y(t))h'(z(t)) \\
&= f'(t)g(t)h(t) + f(t)g'(t)h(t) + f(t)g(t)h'(t).
\end{aligned}$$

18. True or False (and justify your answer): Let f be a real-valued function of 2 or of 3 variables.
    (a) For any vector u, $D_u f = \nabla f \cdot u$.
    Answer: False. This is only true if the vector u has unit length (i.e. $||u|| = 1$).
    (b) At any point p in the domain of f, $\nabla f(p)$ is the direction of greatest change in f, although this change in f in this direction could be a positive change or a negative change (we'd have to check to see which).
    Answer: False. $\nabla f(p)$ is the direction of greatest change in f at p and furthermore f's change in the direction $\nabla f(p)$ is always positive (assuming that f is not constant around p).
    (c) At any point p in the domain of f, $|\nabla f(p)|$ is the maximum value of $D_u f(p)$ as u is allowed to vary over all unit vectors.
    Answer: True. See Stewart p. 795.

(d) If the domain of f is closed and bounded then there is a number $M > 0$ (independent of x and y) for which f is everywhere $< M$.
Answer: True. See Stewart p. 807.

(e) If the domain of f is closed and bounded then f has a maximum and minimum value and furthermore f attains these values.
Answer: True. See Stewart p. 807.

(f) Suppose that $g(x)$ and $h(y)$ are both real-valued functions of real-variables so that $g(x)h(y)$ is real-valued function of 2 real-variables. If both $g(x)$ and $h(y)$ are continuous then so is $g(x)h(y)$.
Answer: True. Let $m(x, y) = xy$ so that m is everywhere continuous. Also note that $(x, y) \mapsto (g(x), h(x))$ is a continuous function since both of its coordinates (i.e. g and h) are continuous functions of (x, y). Since $g(x)h(y) = m(g(x), h(y))$ and since the composition of continuous functions is continuous, it follows that $g(x)h(y)$ is continuous.

19. You are standing above the point $(x, y) = (1, 3)$ on the surface $z = 20 - (2x^2 + y^2)$.

(a) In which direction should you walk to descent fastest?

(b) If you start to move in this direction, what is the slope of your path when you first start to move?
Answer: The gradient is

$$\nabla z = -4x\vec{i} - 2y\vec{j},$$

therefore $\nabla z(1, 3) = -4\vec{i} - 6\vec{j}$. This vector tells us the direction in which the directional derivative is the greatest, so you should go in the direction of $4\vec{i} + 6\vec{j}$.
The slope will be $-||4\vec{i} + 6\vec{j}|| = -\sqrt{52}$.

20. Suppose that f is any differentiable function of one variable. Define V, a function of two variables, by $V(x, t) = f(x + ct)$, where c is a constant. Show that

$$\frac{\partial V}{\partial t} = c\frac{\partial V}{\partial x}$$

Solution: Define $g(x, t) = x + ct$. Then $V = f \circ g$, and the chain rule tells us that

$$\frac{\partial}{\partial t}(f \circ g) = \frac{df}{dg}\frac{\partial g}{\partial t} = c\frac{df}{dg}$$

At the same time,

$$\frac{\partial}{\partial x}(f \circ g) = \frac{df}{dg}\frac{\partial g}{\partial x} = \frac{df}{dg}$$

21. Let $f(x, y) = x^4 + y^4 - 4xy + 1$. Find all (if any) local maximum and minimum values and all saddle points of $f(x, y)$ by doing the following:

(a) Compute f_x and f_y and then find the critical points of $f(x, y)$.

(b) Compute f_{xx}, f_{yy}, f_{xy}, and f_{yx}. Did you need to do two computations to find f_{xy} and f_{yx}?

(c) Compute $D = f_{xx}f_{yy} - [f_{xy}]^2$ and classify all critical points.

Solution: Note that $f_x(x,y) = 4x^3 - 4y$ and $f_y(x,y) = 4y^3 - 4x$. Also, $f_{xx}(x,y) = 12x^2$, $f_{yy}(x,y) = 12y^2$, and $f_{xy}(x,y) = f_{yx}(x,y) = -4$, where the last equality is guaranteed by Clairaut's Theorem (since f is smooth) so that we only needed to compute one of f_{xy} and f_{yx} to get both. This gives us that $D(x,y) = (12xy)^2 - 16$.

The critical points are all (x,y) where $f_x(x,y) = 0$ and $f_y(x,y) = 0$. These equations imply that $0 = 4x^3 - 4y$ and $0 = 4y^3 - 4x$, which become $0 = x^3 - y$ and $0 = y^3 - x$. So $x = y^3 = (x^3)^3 = x^9$, which becomes $0 = x^9 - x = x(x^8 - 1) = x(x^4 - 1)(x^+1) = x(x^2 - 1)(x^2 + 1)(x^4 + 1) = x(x-1)(x+1)(x^2+1)(x^4+1)$. Since $x^2 + 1$ and $x^4 + 1$ are never 0, this implies that $x = 0, x = 1$, or $x = -1$. Using $y = x^3$ gives us the following critical points: $(-1,-1)$, $(0,0)$, $(1,1)$.

At $(0,0)$, the value of D is $D = 0 - 16 = -16 < 0$ so that this is a saddle point.

At both $(-1,-1)$ and $(1,1)$, the value of D is $D = 12^2 - 16 > 0$ and $f_{xx} = 12(\pm 1)^2 = 12 > 0$ so that there is a local minimum at both $(-1,-1)$ and $(1,1)$.