Math 2400: Midterm 2 Review

1. Let $p_0 = (1, 0, 2)$. In 3-dimensional space, identify the range of each of the following as either a line, a plane, or neither:

   (a) $\vec{r_1}(t) = p_0 + t < -2, 0, 1 >$

   (b) $\vec{r_2}(t) = p_0 + t^2 < -2, 0, 1 >$

   (c) $\vec{r_3}(t) = p_0 + t^3 < -2, 0, 1 >$

   (d) $\vec{r_4}(t) =< 1 - 2t + t^2, t, -2t >$

   (e) $\vec{r_5}(s, t) = p_0 + s < 1, -1, 2 > +t < -2, 0, 1 >$

   (f) $\vec{r_6}(s, t) = p_0 + s < -2, 0, 1 > +t < -2, 0, 1 >$

   (g) $\vec{r_7}(s, t) = p_0 + s < 0, 0, 0 > +t < -2, 0, 1 >$

   (h) $\vec{r_8}(s, t) =< 1 - 2s^2 + t, t, 2 + s + 2t^2 >$

2. Let $x(t) = 1 + t$, $y(t) = -t$, and $z(t) = 2 + 2t$. Write this symmetric equations of this line and write the vector equation of this line.

3. Find the domain and derivative/partial derivatives of each of the following:

   (a) $\vec{r_1}(t) = \langle \sqrt{9 - t^2}, \ln(t - 1), e^{t^2} \rangle$

   (b) $\vec{r_2}(t) = \langle \frac{1}{\pi^2 - 4t^2}, \tan t, \arcsin t \rangle$

   (c) $\vec{r_3}(s, t) = \langle \frac{1}{\pi^2 - 4s^2}, \tan s, \arcsin t \rangle$

   (d) $\vec{r_4}(t) = \langle \sin t, \cos t, e^t \rangle$

4. Let $r(t) = \langle e^t \cos t, e^t \sin t, e^t \rangle$. Compute $T(t)$, $N(t)$, $B(t)$, the curvature $\kappa(t)$, and the length of $r(t)$ from a to b.

5. True or False (and justify your answer): Let $f(t)$ be a real-valued function and let $\vec{u}(t)$, $\vec{v}(t)$, and $\vec{r}(t)$ be vector valued functions.

   (a) $\frac{d}{dt}(f(t)\vec{u}(t)) = f'(t)\vec{u}(t) + f(t)\vec{u}'(t)$

   (b) $\frac{d}{dt}(\vec{u}(t) \cdot \vec{v}(t)) = \vec{u}'(t) \cdot \vec{v}'(t)$

   (c) $\frac{d}{dt}(\vec{u}(t) \times \vec{v}(t)) = \vec{u}'(t) \times \vec{v}'(t)$

   (d) If $\vec{r}(t) \neq 0$ then $\frac{d}{dt}|r(t)| = \frac{d}{dt}\sqrt{r(t) \cdot r(t)} = \frac{1}{|r(t)|} r(t) \cdot r'(t)$.

   (e) If $\vec{r}(t) = \langle t, f(t) \rangle$ then the length of $f(t)$ from $t = a$ to $t = b$ is the same as the length of $r(t)$ from a to b.

   (f) Let $T(t)$ be the unit tangent vector of $r(t)$ and let $N(t)$ be the unit normal vector of $r(t)$. Then the unit tangent vector of $T(t)$ is $N(t)$.

   (g) Fix a point p_0 in space that $r(t)$ goes through. Then the curvature of the curve $r(t)$ at p_0 depends on the paramterization of $r(t)$.

   (h) The curvature of $r(t)$ is $\kappa(t) = \frac{|T'(t)|}{|r'(t)|} = \frac{|r'(t) \times r''(t)|}{|r'(t)|^3}$.

(i) It is possible for $T(t)$ to point in the opposite direction of $r'(t)$.

(j) $T(t)$ and $N(t)$ are perpendicular.

(k) If $s(t)$ is the arclength function of $\vec{r}(t)$ (starting at some arbitrary $t = a$) then $\frac{ds}{dt} = |r'(t)|$.

6. Let $\vec{r}(t)$ be a vector valued function. What is wrong with the following reasoning?
The length of $r(t)$ from $t = a$ to $t = b$ is $L = \int_a^b |r'(t)|dt = |\int_a^b r'(t)dt| = |r(t)|_a^b| = |r(b) - r(a)|$.

7. Let $\vec{r}(t)$, $\vec{v}(t)$, and $\vec{a}(t)$ denote the position, velocity, and acceleration of a particle at time t. Suppose that $\vec{r}(0) = \langle 1, \ln 4, 1\rangle$ and $\vec{v}(0) = \langle 1, 1, 0\rangle$. For each of the below, find all three of $\vec{r}(t)$, $\vec{v}(t)$, and $\vec{a}(t)$.

   (a) $\vec{r}(t) = \left\langle e^{\sin t}, \ln((2+t)^2), \cos t\right\rangle$.

   (b) $\vec{v}(t) = \langle \frac{1}{1+t^2}, \frac{2e^t}{1+e^t}, \sin t \cos t\rangle$.

   (c) $\vec{a}(t) = \left\langle e^t, -\frac{1}{(1+t)^2}, -\cos t\right\rangle$.

8. Parameterize the following surfaces:

   (a) The sphere of radius R centered at the origin.

   (b) The sphere of radius R centered at $\vec{p_0} = \langle a, b, c\rangle$.

   (c) The plane that contains the lines $\vec{p_0} + t\vec{v}$ and $\vec{p_0} + t\vec{w}$, where $\vec{p_0} = \langle a, b, c\rangle$, $\vec{v} = \langle v_1, v_2, v_3\rangle$, $\vec{w} = \langle w_1, w_2, w_3\rangle$, and where we assume that $\vec{v}$ and $\vec{w}$ are not parallel.

   (d) Suppose that P is a plane that goes through the origin and has normal vector $\vec{n} = \langle n_1, n_2, n_3\rangle$. Suppose we know that $n_3 \neq 0$. Find equations that parameterize P.

   (e) Suppose that we have given a real-valued function $f(y)$ with domain D, where D is a set of real-numbers. Find a parameterization for the surface of revolution obtained by revolving $f(y)$ around the y-axis.

   (f) What surface does $\vec{r}(r, \theta) = \langle r\cos\theta, r, r\sin\theta\rangle$ parameterize?

   (g) Suppose that $f(y, z)$ is a real-valued function. Find a parameterization for the graph of $f(y, z)$. Apply it to the specific case of $f(y, z) = y^2 + 2z^2$.

9. Draw the level surfaces of $f(x, y) = \sqrt{(x-2)^2 + (y+1)^2}$ and then describe the graph of this function.what is the domain of this function?

10. Draw a contour map of the function $f(x, y) = ye^{-x}$ showing several level curves.

11. State Clairaut's Theorem.

12. True or False (and justify your answer): Let $f(x, y)$ be a real-valued function.

    (a) If $f_x(x, y)$ and $f_y(x, y)$ exist for all x and y, then $f(x, y)$ is continuous.

    (b) If $f_x(x, y)$ and $f_y(x, y)$ exist for all x and y and both are continuous functions then $f(x, y)$ is continuous.

    (c) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2 + y^4}$ exists.

(d) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy^4}{x^2+y^4}$ exists.

(e) Suppose that f is defined on an open disk D. If all of f's partial derivatives exist and are continuous on D then $f_{xyy} = f_{yyx}$ at every point of D.

13. Find the total differential of $z = f(x, y) = x^2 + 3xy - y^2$ and use it to estimate $f(2.05, 2.96)$.

14. Find the tangent plane to $f(x, y) = x^2 + 3xy - y^2$ at $(x, y) = (2, 3)$.

15. Find an equation for the tangent plane to the surface given by

$$xy + yz^2 + z = 0$$

at the point $(-2, 1, 1)$.

16. Suppose that $xyz = \cos(x + y + z)$. *Use implicit differentiation* to find $\frac{\partial y}{\partial x}$ and $\frac{\partial y}{\partial z}$.

17. Let $w = f(t)g(t)h(t)$. Use the chain rule applied to $p(x, y, z) = f(x)g(y)h(z)$ to calculate $\frac{dw}{dt}$.

18. True or False (and justify your answer): Let f be a real-valued function of 2 or of 3 variables.

(a) For any vector u, $D_u f = \nabla f \cdot u$.

(b) At any point p in the domain of f, $\nabla f(p)$ is the direction of greatest change in f, although this change in f in this direction could be a positive change or a negative change (we'd have to check to see which).

(c) At any point p in the domain of f, $|\nabla f(p)|$ is the maximum value of $D_u f(p)$ as u is allowed to vary over all unit vectors.

(d) If the domain of f is closed and bounded then there is a number $M > 0$ (independent of x and y) for which f is everywhere $< M$.

(e) If the domain of f is closed and bounded then f has a maximum and minimum value and furthermore f attains these values.

(f) Suppose that $g(x)$ and $h(y)$ are both real-valued functions of real-variables so that $g(x)h(y)$ is real-valued function of 2 real-variables. If both $g(x)$ and $h(y)$ are continuous then so is $g(x)h(y)$.

19. You are standing above the point $(x, y) = (1, 3)$ on the surface $z = 20 - (2x^2 + y^2)$.

(a) In which direction should you walk to descent fastest?

(b) If you start to move in this direction, what is the slope of your path when you first start to move?

20. Suppose that f is any differentiable function of one variable. Define V, a function of two variables, by $V(x, t) = f(x + ct)$, where c is a constant. Show that

$$\frac{\partial V}{\partial t} = c\frac{\partial V}{\partial x}$$

21. Let $f(x, y) = x^4 + y^4 - 4xy + 1$. Find all (if any) local maximum and minimum values and all saddle points of $f(x, y)$ by doing the following:

(a) Compute f_x and f_y and then find the critical points of $f(x, y)$.

(b) Compute f_{xx}, f_{yy}, f_{xy}, and f_{yx}. Did you need to do two computations to find f_{xy} and f_{yx}?

(c) Compute $D = f_{xx}f_{yy} - [f_{xy}]^2$ and classify all critical points.