Math 2400 Exam 1 Review Solutions

1. Complete the square:

$$x^2+y^2+z^2-6x+4y-2z=8$$

$$\Rightarrow \quad x^2-6x+9+y^2+4y+4+z^2-2z+1=8+9+4+1$$

$$\Rightarrow \quad (x-3)^2+(y+2)^2+(z-1)^2=22.$$

The center of the sphere is $(3,-2,1)$ and the radius is $\sqrt{22}$.

2. (a) $2\vec{a}-\vec{b} = 8\vec{i}+2\vec{j}-(\vec{i}-2\vec{j}+3\vec{k})$

$= 7\vec{i}+4\vec{j}-3\vec{k}$

(b) $\|\vec{a}\| = \sqrt{4^2+1^2} = \sqrt{17}$

(c) $\vec{a}\cdot\vec{b} = 4\cdot 1 + 1(-2) + 0\cdot 3$

$= 2$

(d) Let θ be the angle between $\vec{a}$ and $\vec{b}$.

Then $\cos\theta = \dfrac{\vec{a}\cdot\vec{b}}{\|\vec{a}\|\,\|\vec{b}\|} = \dfrac{2}{\sqrt{17}\,\sqrt{14}} = \dfrac{2}{\sqrt{238}}$

$$\Rightarrow \quad \theta = \arccos\left(\frac{2}{\sqrt{238}}\right).$$

3. Let $\vec{r} = \vec{i} - 4\vec{j} + 2\vec{k}$ be the displacement vector. Then the work done is

$$W = \vec{F} \cdot \vec{r} = 1 \cdot 1 + (-6)(-4) + 2 \cdot 2$$

$$= 29 \text{ N} \cdot \text{m}$$

4. (a) The distance from P to Q is

$$\sqrt{(4-3)^2 + (0-(-2))^2 + (1-0)^2} = \sqrt{1^2 + 2^2 + 1^2}$$

$$= \sqrt{6}.$$

(b) The area of the triangle with vertices P, Q, and R is $\frac{1}{2} \| \vec{PQ} \times \vec{PR} \|$.

$$\vec{PQ} \times \vec{PR} = \langle 1, 2, 1 \rangle \times \langle -2, 4, 1 \rangle$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & 1 \\ -2 & 4 & 1 \end{vmatrix}$$

$$= \langle -2, -3, 8 \rangle.$$

$\Rightarrow$ The area is $\frac{1}{2} \sqrt{(-2)^2 + (-3)^2 + 8^2} = \dfrac{\sqrt{77}}{2}.$

(c) The vector $\vec{PQ} \times \vec{PR} = \langle -2, -3, 8 \rangle$ is normal to the plane containing P, Q, and R. So the equation of the plane is

$$-2x - 3y + 8z = d,$$

for some constant d. Plug in the coordinates of P to find d:

$$d = -2(3) - 3(-2) + 8(0) = 0.$$

5. Let $\vec{a} = \langle a_1, a_2, a_3 \rangle$, $\vec{b} = \langle b_1, b_2, b_3 \rangle$ and $\vec{c} = \langle c_1, c_2, c_3 \rangle$. Then

$$\vec{a} \times (\vec{b} \times \vec{c}) = \vec{a} \times \langle b_2c_3 - b_3c_2, b_3c_1 - b_1c_3, b_1c_2 - b_2c_1 \rangle$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_2c_3 - b_3c_2 & b_3c_1 - b_1c_3 & b_1c_2 - b_2c_1 \end{vmatrix}$$

$$= (a_2(b_1c_2 - b_2c_1) - a_3(b_3c_1 - b_1c_3))\vec{i}$$

$$+ (a_3(b_2c_3 - b_3c_2) - a_1(b_1c_2 - b_2c_1))\vec{j}$$

$$+ (a_1(b_3c_1 - b_1c_3) - a_2(b_2c_3 - b_3c_2))\vec{k}.$$

Consider the first component of this vector. Rearranging gives:

$$a_2(b_1c_2 - b_2c_1) - a_3(b_3c_1 - b_1c_3)$$

$$= b_1(a_2c_2 + a_3c_3) - c_1(a_2b_2 + a_3b_3)$$

$$= b_1(a_2c_2 + a_3c_3) + a_1b_1c_1 - a_1b_1c_1 - c_1(a_2b_2 + a_3b_3)$$

$$= b_1(a_1c_1 + a_2c_2 + a_3c_3) - c_1(a_1b_1 + a_2b_2 + a_3b_3)$$

$$= b_1(\vec{a}\cdot\vec{c}) - c_1(\vec{a}\cdot\vec{b}).$$

By a similar argument, the second and third components of $\vec{a} \times (\vec{b} \times \vec{c})$ are $b_2(\vec{a}\cdot\vec{c}) - c_2(\vec{a}\cdot\vec{b})$ and $b_3(\vec{a}\cdot\vec{c}) - c_3(\vec{a}\cdot\vec{b})$, respectively. Therefore,

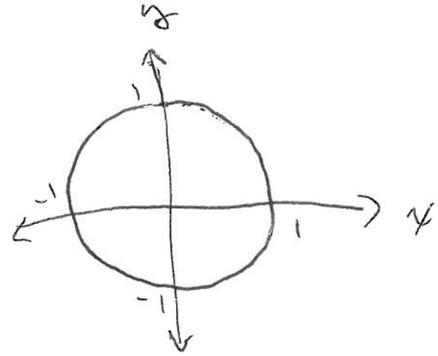
$$\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c}.$$

6. (a) The domain of f is $\mathbb{R}^2$. The range of f is $(-\infty, 2]$.

(b) $1 = 2 - \sqrt{x^2+y^2}$

$z = 1$: $\Rightarrow \sqrt{x^2+y^2} = 1$

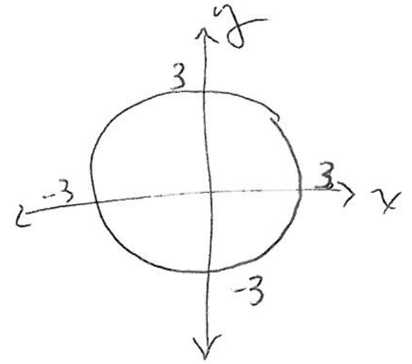
$\Rightarrow x^2+y^2 = 1$



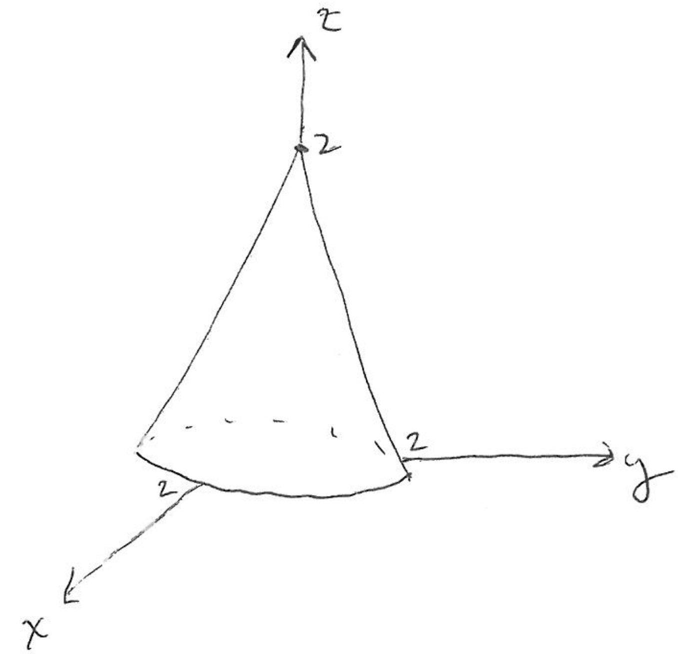
$z = -1$: $-1 = 2 - \sqrt{x^2+y^2}$

$\Rightarrow \sqrt{x^2+y^2} = 3$

$\Rightarrow x^2+y^2 = 9$



(c)



(d) $2 - \sqrt{x^2+y^2} = 2 - \sqrt{r^2} = 2 - |r|$

7. The line we're looking for is perpendicular to both $\langle 1,1,1\rangle$ (normal vector for the plane $x+y+z=2$) and $\langle 1,-1,2\rangle$ (direction vector for the line $x=1+t, y=1-t, z=2t$). So

$$\langle 1,1,1\rangle \times \langle 1,-1,2\rangle = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & 1 \\ 1 & -1 & 2 \end{vmatrix} = \langle 3,-1,-2\rangle$$

points in the direction of the line. Since

$(0,1,2)$ is a point on the line, parametric equations for the line are

$$x = 3t$$
$$y = 1 + t$$
$$z = 2 - 2t.$$

8. (a) ellipsoid
   (b) hyperboloid of one sheet
   (c) hyperbolic paraboloid
   (d) cone

9. (a)

$$\begin{aligned} x &= \rho \sin\varphi \cos\theta \\ &= 2 \sin\left(\tfrac{\pi}{2}\right) \cos(\pi) \\ &= -2 \end{aligned}$$

$$\begin{aligned} y &= \rho \sin\varphi \sin\theta \\ &= 2 \sin\left(\tfrac{\pi}{2}\right) \sin(\pi) \\ &= 0 \end{aligned}$$

$$\begin{aligned} z &= \rho \cos\varphi \\ &= 2 \cos\left(\tfrac{\pi}{2}\right) \\ &= 0. \end{aligned}$$

(b) $x = 3\sin\left(\frac{3\pi}{2}\right)\cos(0)$

$= -3$

$y = 3\sin\left(\frac{3\pi}{2}\right)\sin(0)$

$= 0$

$z = 3\cos\left(\frac{3\pi}{2}\right)$

$= 0$.

10. (a) cylinder of radius 3 centered on the z-axis

(b) vertical half-plane making an angle of $\frac{\pi}{3}$ with the positive x-axis

(c) half-cone making an angle of $\frac{\pi}{3}$ with the positive z-axis