Second Midterm Solutions

October 28, 2013

1. (a) By the definition of the partial derivatives we have

$$f_x(0,0) = \lim_{h\to 0} \frac{f((0,0)+h(1,0)}{h} = \lim_{h\to 0} \frac{f(h,0)}{h} = \lim_{h\to 0} \frac{\sqrt{|h\cdot 0|}+h}{h} = \lim_{h\to 0} \frac{h}{h} = 1$$

and

$$f_y(0,0) = \lim_{h\to 0} \frac{f((0,0)+h(0,1)}{h} = \lim_{h\to 0} \frac{f(0,h)}{h} = \lim_{h\to 0} \frac{\sqrt{|0\cdot h|}+0}{h} = \lim_{h\to 0} \frac{0}{h} = 0.$$

(b) Assuming h to be positive

$$f_{\vec{u}}(0,0) = \lim_{h\to 0} \frac{f(h/\sqrt{2}, h/\sqrt{2}) - f(0,0)}{h} = \lim_{h\to 0} \frac{\sqrt{h^2/2}+h/\sqrt{2}}{h} = \lim_{h\to 0} \frac{2h/\sqrt{2}}{h} = \frac{2}{\sqrt{2}}.$$

(c) Using the results of part (a), we have $(f_x(0,0)\vec{i} + f_y(0,0)\vec{j}) \cdot \vec{u} = \vec{i} \cdot (\frac{1}{\sqrt{2}}\vec{i} + \frac{1}{\sqrt{2}}\vec{j}) = \frac{1}{\sqrt{2}}$.

2. (a) The extreme value theorem does not apply to this problem as given, because the region restricting the dimensions $(x, y, z > 0)$ is not closed and bounded.

(b) We want to minimize the cost function, which, after drawing a picture and labeling side lengths x, y, and z is given by:

$$C(x,y,z) = 6xy + 4xz + 2yz.$$

The constraint that the volume is 60 gives that $xyz = 60$, or, because none of the dimensions can be 0,

$$z = \frac{60}{xy}.$$

Eliminating z from the cost equation gives

$$C(x,y) = 6xy + \frac{240}{y} + \frac{120}{x}.$$

To find critical points we solve the system

$$C_x = 6y - \frac{120}{x^2} = 0 \Leftrightarrow 6yx^2 - 120 = 0 \iff 2yx^2 = 40$$
$$C_y = 6x - \frac{240}{y^2} = 0 \Leftrightarrow 6xy^2 - 240 = 0 \iff xy^2 = 40$$

Using substitution we have

$$\begin{aligned} xy^2 &= 2yx^2 \\ 0 &= 2yx^2 - xy^2 \\ 0 &= xy(2x-y) \text{ (and } x \neq 0 \neq y) \\ 0 &= 2x - y \\ 2x &= y \end{aligned}$$

Then setting $C_x = 0$ gives $4x^3 = 40$, so $x = \sqrt[3]{10}$. Then $y = 2\sqrt[3]{10}$. Finally,

$$z = \frac{60}{2\sqrt[3]{10}\sqrt[3]{10}} = \frac{30}{10^{\frac{2}{3}}} = 3\sqrt[3]{10}.$$

To see if this gives a local minimum we find D.
First some more derivatives: $C_{xx} = \frac{240}{x^3}$, $C_{yy} = \frac{480}{y^3}$, and $C_{xy} = 6$.
Then

$$D(\sqrt[3]{10}, 2\sqrt[3]{10}) = \frac{240}{10} \cdot \frac{480}{20} - 36 = 24 \cdot 24 - 36 = 540 > 0,$$

so the critical point is either a max or a min. Since $f_{xx}(\sqrt[3]{10}) = 24 > 0$, this is a local min.

3. (a) By the multivariable chain rule,

$$h'(t) = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} = \nabla f(x, y) \cdot (x'(t)\vec{i} + y'(t)\vec{j}).$$

(b) Differentiating both sides with respect to t, and using part (a),

$$\nabla f(x, y) \cdot (x'(t)\vec{i} + y'(t)\vec{j}) = 0.$$

This implies that $\nabla f(x, y)$ and $x'(t)\vec{i} + y'(t)\vec{j}$ are perpendicular for all t.

4. We have critical points whenever $\nabla f = 0$, that is, where simultaneously

$$\begin{aligned} f_x(x, y) &= 3x^2 - 6x = 0, \text{ and} \\ f_y(x, y) &= 2y + 10 = 0 \end{aligned}$$

Solving $f_y = 0$ gives $y = -5$. Solving $3x^2 - 6x = 0$ gives $x = 0$ or $x = 2$. Thus we have two critical points: $(0, 5)$ and $(2, 5)$.

To classify these points we use the second derivative test, a.k.a. the discriminant,

$$D(x, y) = f_{xx}(x, y) \cdot f_{yy}(x, y) - (f_{xy}(x, y))^2.$$

First we compute

$$\begin{aligned} f_{xx} &= 6x - 6, \\ f_{yy} &= 2, \text{ and} \\ f_{xy} &= 0 \end{aligned}$$

Plugging in gives:

$$\begin{aligned} D(0, 5) &= (-6)(2) - 0 \\ &= -12, \\ \text{and } D(2, 5) &= (6)(2) - 0 \\ &= 12 \end{aligned}$$

Because $D(0, 5) < 0$, the point $(0, 5)$ must be a saddle point. Because $D(2, 5) > 0$, the point $(2, 5)$ must be either a max or a min. We inspect $f_{xx}(2, 5) = 6 > 0$, and conclude that $(2, 5)$ is a min.

5. (a) The slope of the steepest path up the hill at the point $(5, 10, 1150)$ is the magnitude of the gradient. As

$$\frac{\partial z}{\partial x} = -4x \quad \text{and} \quad \frac{\partial z}{\partial y} = -6y,$$

we get $\nabla f(5, 10) = \langle -20, -60 \rangle$. Thus $\|\nabla f(5, 10)\| = \sqrt{4000}$, which is the slope of the steepest path up the hill.

(b) The gradient at $(5, 10)$ is perpendicular to the contour $z = 1150$. This means that any vector perpendicular to $\nabla f(5, 10)$ will be parallel to the contour. One such vector is $\langle 60, -20\rangle$, and this vector points in the clockwise direction of the contour. To make this a unit vector, we divide by the magnitude of the vector, and

$$\vec{u} = \frac{1}{\sqrt{4000}}\langle 60, -20\rangle$$

is the desired unit vector.

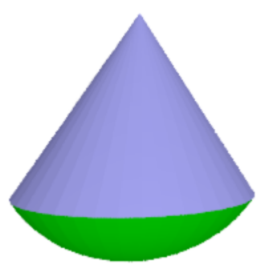
6. (a) The cone $z = 8 - \sqrt{3(x^2 + y^2)}$ opens downwards and has tip at $(0, 0, 8)$. The sphere has radius 4 and is centered at $(0, 0, 4)$. First we will find where the cone and the sphere intersect. We can rewrite the equation of the cone as

$$z - 4 = 4 - \sqrt{3(x^2 + y^2)}$$

and substituting this into the equation of the sphere

$$\begin{aligned} x^2 + y^2 + (4 - \sqrt{3(x^2 + y^2)})^2 &= 16 \\ x^2 + y^2 + 16 - 8\sqrt{3(x^2 + y^2)} + 3(x^2 + y^2) &= 16 \\ 4(x^2 + y^2) - 8\sqrt{3}\sqrt{x^2 + y^2} &= 0 \\ \sqrt{x^2 + y^2}(\sqrt{x^2 + y^2} - 2\sqrt{3}) &= 0 \end{aligned}$$

which implies $(x, y) = (0, 0)$ and $x^2 + y^2 = 12$ are solutions. By plugging these solutions into either the equation for the sphere or cone, we see that the former solution occurs at $z = 8$ so that the sphere and cone intersect at the single point $(0, 0, 8)$. The latter solution occurs at $z = 2$ so that the sphere and cone intersect in a circle centered at the origin of radius $2\sqrt{3}$ in the $z = 2$ plane. A picture of the solid is shown below.



We will write a triple integral in Cartesian coordinates integrating with respect to z first, then y, then x. As the circle of intersection occurs in the $z = 2$ plane, which is 2 units below the center of the sphere, the lower hemisphere forms the lower boundary of the solid. In order to find a lower limit of integration, we write the equation for the sphere as a function of x and y by solving for z:

$$\begin{aligned} x^2 + y^2 + (z - 4)^2 &= 16 \\ (z - 4)^2 &= 16 - x^2 - y^2 \\ z - 4 &= \pm\sqrt{16 - x^2 - y^2} \end{aligned}$$

and because the lower hemisphere is the lower boundary we consider only the solution

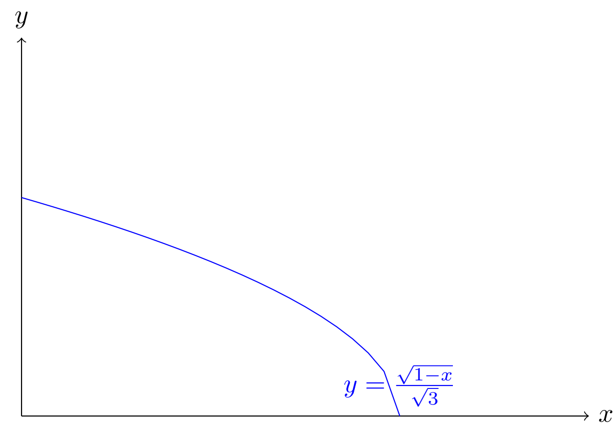
$$\begin{aligned} z - 4 &= -\sqrt{16 - x^2 - y^2} \\ z &= 4 - \sqrt{16 - x^2 - y^2} \end{aligned}$$

Also, as the projection of the solid is the disk $x^2 + y^2 \leq 12$ in the xy-plane, the y-limits of integration are given by $y = -\sqrt{12 - x^2}$ for the lower limit and $y = \sqrt{12 - x^2}$ and the x-limits of integration are given by $x = -2\sqrt{3}$ and $x = 2\sqrt{3}$.

Hence an integral describing the volume of the solid S is

$$V = \iiint_S dV = \int_{x=-2\sqrt{3}}^{2\sqrt{3}} \int_{y=-\sqrt{12-x^2}}^{\sqrt{12-x^2}} \int_{z=4-\sqrt{16-x^2-y^2}}^{8-\sqrt{3(x^2+y^2)}} dz\, dy\, dx$$

(b) i. The region is given by



ii. We evaluate by changing the order of integration. Since the top y bound is $y = \frac{\sqrt{1-x}}{\sqrt{3}}$, solving for x in terms of y, we find that the top x bound is $x = 1-3y^2$. Looking at the picture above we see that the lower x bound is 0. Since $y = \frac{1}{\sqrt{3}}$ when $x = 0$ on the curve, the top y bound is $\frac{1}{\sqrt{3}}$, and looking at the picture above we find that the lower y bound is 0. Therefore we can rewrite our integral as

$$\int_0^{\frac{1}{\sqrt{3}}} \int_0^{1-3y^2} e^{-y^3+y}\, dxdy$$

Integrating with respect to x we get

$$\int_0^{\frac{1}{\sqrt{3}}} xe^{-y^3+y}|_0^{1-3y^2}\, dy = \int_0^{\frac{1}{\sqrt{3}}} (1-3y^2)e^{-y^3+y}\, dy$$

Now, either recognizing that $1-3y^2$ is the derivative of $-y^3+y$ or by u substitution with $u = -y^3+y$ we get

$$e^{-y^3+y}|_0^{\frac{1}{\sqrt{3}}} = e^{-(\frac{1}{\sqrt{3}})^3+\frac{1}{\sqrt{3}}} - 1$$