

Math 4001 Analysis 2
Homework Set 6

Fall 2023

Course Instructor: Dr. Markus Pflaum

Contact Info: Office: Math 255, Telephone: 2-7717, e-mail: markus.pflaum@colorado.edu.

Problem 1: Prove that the function

$$F : \mathbb{R}^n \times \mathbb{R}_{>0} \rightarrow \mathbb{R}, \quad (x, t) \mapsto t^{-n/2} \exp(-\|x\|^2/4t)$$

satisfies the differential equation

$$\Delta F - \frac{\partial F}{\partial t} = 0,$$

where $\Delta = \frac{\partial^2}{\partial x_1^2} + \dots + \frac{\partial^2}{\partial x_n^2}$ is the so-called Laplacian on \mathbb{R}^n . (6P)

Hint: You may use without proof the fact that the function

$$g : \mathbb{R} \rightarrow \mathbb{R}, \quad t \mapsto \begin{cases} 0 & \text{for } t = 0, \\ e^{-\frac{1}{t^2}} & \text{for } t \neq 0 \end{cases}$$

is smooth on \mathbb{R} .

For the proof of the hint you get the following extra points: (4P)

Problem 2: Compute the divergence of the vector field

$$F : \mathbb{R}^n \setminus \{0\} \rightarrow \mathbb{R}^n, \quad x \mapsto \frac{x}{\sqrt{x_1^2 + \dots + x_n^2}}.$$

Hint: The divergence of a differentiable function $F : U \rightarrow \mathbb{R}^n$ with $U \subset \mathbb{R}^n$ open is defined to be the function

$$\operatorname{div} F := \sum_{i=1}^n \frac{\partial f_i}{\partial x_i} : U \rightarrow \mathbb{R}. \quad (4P)$$

Problem 3: Determine the Taylor expansion of the function

$$h : \mathbb{R}_{>0} \times \mathbb{R}_{>0} \rightarrow \mathbb{R}, \quad h(x, y) = \frac{x - y}{x + y}$$

at the point $(1, 1)$ up to order 2. (6P)

Problem 4: Let $f, g : \Omega \rightarrow \mathbb{R}$ be two differentiable functions, where $\Omega \subset \mathbb{R}^n$ is open. Prove that the product $f \cdot g$ is differentiable, and determine a formula for the Jacobian of $f \cdot g$. (4P)