

Math 4001/5001 Analysis 2
Homework Set 5

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Problem 1: Consider the series $\sum_{n=1}^{\infty} (-1)^n t^n$. Determine its convergence radius and limit on the convergence circle. Use this to show via integration that for real x with $|x| < 1$

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n .$$

(4P)

Problem 2: Using that $z \mapsto \ln(1+z) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} z^n$ is well-defined for all complex z with $|z| < 1$ define for all $\alpha \in \mathbb{C}$ the function

$$(1+\cdot)^\alpha : B(0,1) \mapsto \mathbb{C}, \quad z \mapsto (1+z)^\alpha := e^{\alpha \ln(1+z)} .$$

Determine the (complex) derivative of this function and show that for all $\alpha, \beta \in \mathbb{C}$

$$(1+z)^{\alpha+\beta} = (1+z)^\alpha \cdot (1+z)^\beta .$$

(4P)

Problem 3: For $\alpha \in \mathbb{R}$ consider the power series

$$\sum_{n=0}^{\infty} \binom{\alpha}{n} z^n ,$$

where $\binom{\alpha}{n} = \prod_{k=1}^n \frac{\alpha-k+1}{k}$. Determine its convergence radius R and denote the resulting function on $B(0, R)$ by f that is

$$f(z) = \sum_{n=0}^{\infty} \binom{\alpha}{n} z^n \quad \text{for } z \in B(0, R) .$$

Prove that $f(z) = (1+z)^\alpha$ by showing that

$$f'(z) = \frac{\alpha}{1+z} f(z)$$

and deriving from this that $g(z) := f(z)(1+z)^{-\alpha}$ is constant 1. (6P)

Problem 4: Let $A \in \mathfrak{M}(n \times n, \mathbb{R})$ be a real symmetric matrix, and $q : \mathbb{R}^n \rightarrow \mathbb{R}$ the map $x \mapsto \langle x, Ax \rangle$. It is called the *quadratic form* associated to the symmetric matrix A . Compute the derivative $Dq(x)$ of q at every $x \in \mathbb{R}^n$. (4P)