Math 4001/5001 Analysis 2
Homework Set 4

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Problem 1: Show that the map

$$f : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}^n, \quad x \mapsto \frac{x}{x_1^2 + \ldots + x_n^2}$$

is differentiable at each point $x \in \mathbb{R}^n \setminus \{0\}$, and determine its derivative by computing the Jacobian at each point. Is it possible to extend f to a continuous function on $\mathbb{R}^n$? (4P)

Problem 2: Let

$$g : \mathbb{R}^2 \to \mathbb{R}, \quad (x, y) \mapsto \begin{cases} 0, & \text{if } (x, y) = (0, 0), \\ \frac{x^3}{x^2+y^2} & \text{if } (x, y) \neq (0, 0). \end{cases}$$

Prove that g is everywhere partially differentiable on its domain. Determine all partial derivatives. Where is g differentiable? (4P)

Problem 3: Compute the Jacobian of the function

$$f : \mathbb{R}^3 \to \mathbb{R}^3, \quad (r, \theta, \varphi) \mapsto (r \sin\theta \cos\varphi, r \sin\theta \sin\varphi, r \cos\theta).$$

(4P)

Problem 4: Define $g : \mathbb{R}^2 \to \mathbb{R}$ by

$$g(x, y) := \begin{cases} xy\frac{x^2-y^2}{x^2+y^2}, & \text{for } (x, y) \neq (0, 0), \\ 0, & \text{for } (x, y) = (0, 0). \end{cases}$$

Prove that g is twice partially differentiable, but that

$$\frac{\partial}{\partial x}\frac{\partial}{\partial y} g(0, 0) \neq \frac{\partial}{\partial y}\frac{\partial}{\partial x} g(0, 0).$$

Is g continuous at the origin? (6P)