Math 4001/5001 Analysis 2
Homework Set 3

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Problem 1:

(a) Let (X,d) be a metric space, and $(A_i)_{i\in I}$ a family of subsets of X. Prove that if each of the A_i is connected and if $A_i \cap A_j \neq \emptyset$ for all $i, j \in I$, then $A = \bigcup_{i\in I} A_i$ is connected. (4P)

(b) Using (a) show that $\mathbb{R}$ is connected. Is $\mathbb{Q}$ connected? (2P)

Problem 2: Let $f : X \to Y$ be a continuous map between metric spaces (X,d) and (Y,ϱ) and assume that X is compact. Prove that then f is *uniformly continuous* which means that for each $\varepsilon > 0$ there exists $\delta > 0$ such that for $x, y \in X$ with $d(x,y) < \delta$ the relation $\varrho\big(f(x), f(y)\big) < \varepsilon$ holds true. (4P)

Problem 3: Verify the Lemma by Lebesgue:
Let (X,d) be a metric space, $K \subset X$ a compact subset, and $(U_i)_{i\in I}$ an open covering of K. Then there exists a number $\lambda > 0$, called a *Lebesgue* number, such that for each subset $A \subset K$ with $\mathrm{diam}(A) := \sup\{d(x,y) \mid x, y \in A\} \leq \lambda$ there exists $i \in I$ with $A \subset U_i$. (6P)

Problem 4: Let f, g be two real-valued continuous functions on a metric space (X,d). Show that then the functions

$$\varphi : X \to \mathbb{R},\ x \mapsto \varphi(x) = \max\{f(x), g(x)\} \quad \text{and} \quad \psi : X \to \mathbb{R},\ x \mapsto \psi(x) = \min\{f(x), g(x)\}$$

are continuous. (4P)