Math 4001/5001 Analysis 2
Homework Set 2

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Problem 1: Let $(X_1, d_1), \ldots, (X_n, d_n)$ be metric spaces, and $X = X_1 \times \ldots \times X_n$ their cartesian products.

a) Show that
$$d_\infty : X \times X \to \mathbb{R}_{\geq 0},\ (x, y) \mapsto \max\{d(x_1, y_1), \ldots, d(x_n, y_n)\}$$
and
$$d_2 : X \times X \to \mathbb{R}_{\geq 0},\ (x, y) \mapsto \sqrt{d_1(x_1, y_1)^2 + \ldots + d_n(x_n, y_n)^2}$$
are metrics on X. (4P)

b) Show that the two metrics d_∞ and d_2 on X are *equivalent* which means that there are $c, C > 0$ such that
$$c\, d_\infty(x, y) \leq d_2(x, y) \leq C\, d_\infty(x, y)$$
for all $x, y \in X$. What does this imply for the topologies associated to d_∞ and d_2? (4P)

Problem 2: Let $I = [a, b] \subset \mathbb{R}$ be a compact interval and $\mathcal{C}^1\big([a, b]\big)$ the space of continuously differentiable functions on I that is of all continuous maps $f : I \to \mathbb{R}$ which are differentiable on the interior (a, b) such that $f' : (a, b) \to \mathbb{R}$ has a continuous extension to I. Show that
$$|||\cdot||| : \mathcal{C}^1\big([a, b]\big) \to \mathbb{R}_{\geq 0}, \quad f \mapsto |||f||| := \sup\{|f(x)| \mid x \in I\} + \sup\{|f'(x)| \mid x \in (a, b)\}$$
is a norm on $\mathcal{C}^1\big([a, b]\big)$. Is $\mathcal{C}^1\big([a, b]\big)$ with this norm a Banach space? (6P)

Problem 3: Let $\mathcal{O}$ be a topology on a set X and $Y \subset X$ a subset. Prove that then
$$\mathcal{O}_Y = \{Y \cap O \mid O \in \mathcal{O}\}$$
is a topology on Y which is called the *induced topology.* If X is a metric space with metric d and $\mathcal{O}$ the corresponding metric topology, how does $\mathcal{O}_Y$ relate to the metric topology on Y where the metric on Y is obtained by restriction of the metric d to Y? (3P)

Problem 4: Let (X, d) be a complete metric space, and $Y \subset X$ a subset. Show that then (Y, d) is complete if and only if Y is closed in X. (3P)