Math 4001/5001 Analysis 2
Homework Set 1

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Problem 1:

a) Write down the axioms of a field. (2P)

b) Prove that $\mathbb{C} \cong \mathbb{R}^2$ together with addition $+$, multiplication $\cdot$ and the elements $0 = (0,0)$ and $1 = (1,0)$ becomes a field. (4P)

c) Show that there is no order relation on $\mathbb{C}$ so that it becomes an ordered field. (2P)

Problem 2: Show that the absolute value $|\cdot| : \mathbb{C} \to \mathbb{R}_{\geq 0}$, $a + ib \mapsto \sqrt{a^2 + b^2}$ is a norm on $\mathbb{C}$. (2P)

Problem 3:

a) Prove that for all real $p \geq 1$ and $x, y \in \mathbb{R}^n$ the following inequality by Minkowski holds true:

$$\|x + y\|_p \leq \|x\|_p + \|y\|_p \ ,$$

where $\|x\|_p = \left(\sum_{k=1}^n |x_k|^p \right)^{\frac{1}{p}}$.
Hint: First show

$$\|x + y\|_p^p = \sum_{k=1}^n |x_k + y_k|^p \leq \sum_{k=1}^n |x_k| \, |x_k + y_k|^{p-1} + |y_k| \, |x_k + y_k|^{p-1}$$

and then use Hölder's inequality which says that for $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$ the following holds true:

$$\sum_{k=1}^n |x_k y_k| \leq \|x\|_p \cdot \|y\|_q$$

(4P)

b) Show that $\|\cdot\|_p$ is a norm on $\mathbb{R}^n$. (2P)

Problem 4: Prove that $\mathbb{R}^n$ with the euclidean norm and $\mathbb{C}$ with the absolute value are complete. (4P)