

Abelian congruences in locally finite Taylor varieties

Tutorial – Lecture 3

Ross Willard

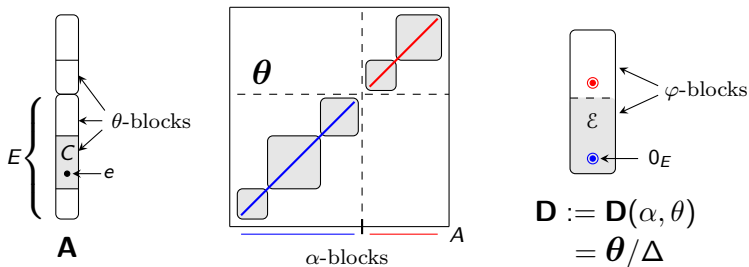
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Recap

Suppose $\mathbf{A} \in \mathcal{V}$ with a WDT, $\theta \leq \alpha$ in $\text{Con } \mathbf{A}$, $[\alpha, \theta] = 0$, and $\Delta := \Delta_{\theta, \alpha}$ the smallest witness.

Let $\mathbf{D} = \mathbf{D}(\alpha, \theta) := \boldsymbol{\theta}/\Delta$ and $\varphi := \bar{\alpha}/\Delta$.



θ -blocks and φ -blocks support abelian groups: $\text{Grp}(\theta, e)$, $\text{Grp}(\varphi, 0_E)$.

If E is an α -block and \mathcal{E} is the corresponding φ -block, then we have embeddings $\chi_e : \text{Grp}(\theta, e) \hookrightarrow \text{Grp}(\varphi, 0_E)$ from each θ -block in E .

Recap (continued)

When \mathbf{A} is finite and $0 \prec \theta$, \exists a finite field \mathbb{F} such that:

- ① Each θ -block naturally inherits the structure of a vector space over \mathbb{F} .
- ② For each $\alpha \geq \theta$ with $[\alpha, \theta] = 0$:
 - (a) Each φ -block of the difference algebra $\mathbf{D}(\alpha, \theta)$ also inherits the structure of a vector space over \mathbb{F} .
 - (b) Each $\chi_e : \text{Grp}(\theta, e) \hookrightarrow \text{Grp}(\varphi, 0_E)$ is an \mathbb{F} -vector space embedding.
- ③ \mathbb{F} and its actions on θ -blocks can be defined (up to isomorphism) from $(\mathbf{D}(\alpha, \theta), \varphi, T_\alpha)$, for any $\alpha \geq \theta$ satisfying $[\alpha, \theta] = 0$.
- ④ \mathbb{F} is determined by the isomorphism type of $(\mathbf{D}(\alpha, \theta), \varphi, T_\alpha)$.

Problem from 2nd lecture:

Why didn't I just let $\alpha = \theta$ and use $\mathbf{D}(\theta, \theta)$ and $\mathbb{F} = \mathbb{F}_{(\mathbf{D}(\theta, \theta), \varphi, \tau_\theta)}$?

Answer: It will be more useful to use the largest α satisfying $[\alpha, \theta] = 0$.

- Called the **annihilator** of θ ; denoted $(0 : \theta)$.
- Cute Fact:

$$\left(0 \prec \theta \text{ and } \alpha = (0 : \theta) \right) \implies \mathbf{D}(\alpha, \theta) \text{ is subdirectly irreducible (SI)} \\ \text{(with monolith } \varphi)$$

Consequence of the Cute Fact

Suppose \mathbf{A}, \mathbf{A}' are finite algebras in a WDT variety, with

$\theta \in \text{Con } \mathbf{A}$ and $\theta' \in \text{Con } \mathbf{A}'$, both minimal and abelian.

$\alpha = (0 : \theta)$ and $\alpha' = (0 : \theta')$.

$\mathbf{D} = \mathbf{D}(\alpha, \theta)$ with monolith φ , $\mathbf{D}' = \mathbf{D}(\alpha', \theta')$ with monolith φ' .

\mathbb{F} = the finite field for θ , and \mathbb{F}' = the finite field for θ' .

Lemma 8

In this situation: if $\mathbf{D}(\alpha, \theta) \cong \mathbf{D}(\alpha', \theta')$, then $\mathbb{F} \cong \mathbb{F}'$.

Proof sketch:

$$\mathbf{D} \cong \mathbf{D}' \Rightarrow (\mathbf{D}, \varphi) \cong (\mathbf{D}', \varphi') \Rightarrow (\mathbf{D}, \varphi, T_\alpha) \cong (\mathbf{D}', \varphi', T_{\alpha'}) \Rightarrow \mathbb{F} \cong \mathbb{F}'.$$

because SI transversal subuniverses are conjugate

Similarity

Definition (Freese 1983)

Suppose \mathbf{A}, \mathbf{A}' are SI algebras in a congruence modular variety, with abelian monoliths μ, μ' respectively.

Say that \mathbf{A} and \mathbf{A}' are *similar* and write $\mathbf{A} \sim \mathbf{A}'$ if (some complicated shit).

Freese also proved:

$$\begin{aligned} \text{(same complicated shit)} \iff \mathbf{D}(\alpha, \mu) \cong \mathbf{D}(\alpha', \mu') \quad \text{where} \\ \alpha = (0 : \mu) \text{ and } \alpha' = (0 : \mu'). \end{aligned}$$

Use this to define \sim between SI's with abelian monoliths in WDT varieties.

Corollary 3

If \mathbf{A}, \mathbf{A}' are finite SI algebras in a WDT variety, with abelian monoliths and respective fields \mathbb{F}, \mathbb{F}' , then $\mathbf{A} \sim \mathbf{A}' \implies \mathbb{F} \cong \mathbb{F}'$.

(Deep breath)

Critical, completely functional relations

Definition

Suppose $\mathbf{A}_1, \dots, \mathbf{A}_n$ are finite algebras in a common signature.

Let $R \leq_{sd} \mathbf{A}_1 \times \dots \times \mathbf{A}_n$.

R is **completely functional** if each coordinate is determined by (i.e., is a function of) the remaining coordinates:

$$\forall i \in [n], \quad \left. \begin{array}{l} (\mathbf{r}, a, \mathbf{s}) \in R \\ (\mathbf{r}, b, \mathbf{s}) \in R \end{array} \right\} \implies a = b.$$

\uparrow
 i

$n = 2$: R completely functional $\iff R$ is the graph of some $\mathbf{A}_1 \cong \mathbf{A}_2$.
Boring

Examples with $n = 3$.

① $\{(a, a, a) : a \in A\} \leq_{sd} \mathbf{A} \times \mathbf{A} \times \mathbf{A}$.

Also boring.

- ② Let $\mathbf{A} = (A, +, 0)$ be a finite abelian group,

$$R_+ := \{(a_1, a_2, a_3) \in A^3 : a_1 + a_2 + a_3 = 0\} \leq_{sd} \mathbf{A} \times \mathbf{A} \times \mathbf{A}.$$

Not boring!

Definition (Kearnes & Szendrei 2012; Zhuk “Key relations” 2017)

Suppose $R \leq_{sd} \mathbf{A}_1 \times \cdots \times \mathbf{A}_n$. R is **critical** if

- ① R is meet-irreducible in the lattice $\text{Sub}(\mathbf{A}_1 \times \cdots \times \mathbf{A}_n)$.
- ② R has no dummy coordinates. (*) In particular, $|A_i| > 1$ for each i .

Examples:

- ① $\{(a, a, a) : a \in A\} \leq_{sd} \mathbf{A} \times \mathbf{A} \times \mathbf{A}$ Not critical
- ② $R_+ \leq_{sd} \mathbf{A} \times \mathbf{A} \times \mathbf{A}$ Critical when \mathbf{A} is cyclic of prime power order.

(*) $R \neq A_1 \times \text{proj}_{\{2, \dots, n\}}(R)$, and similarly for the other coordinates.

Rectangular relations

Another kind of relation:

rectangular (a.k.a. “having the $(1, n-1)$ -parallelogram property”).

I won't define them. But let

$CCF = \{\text{all critical, completely functional relations}\}$

$CR = \{\text{all critical, rectangular relations}\}$

$PCCF = \{\text{all pullbacks of critical, completely functional relations
on quotients of the factor algebras } \mathbf{A}_1, \dots, \mathbf{A}_n\}.$

Then

$$CCF \subseteq CR \subseteq PCCF.$$

Understanding CR (in a locally finite variety) is sometimes important.

Understanding CCF is typically enough.

A worthy goal:

To describe critical rectangular relations in locally finite WDT varieties^{*}.

We focus on critical, completely functional relations of arity ≥ 3 in locally finite WDT varieties.

Building on

- Kearnes & Szendrei (2012) in CM case.
- Zhuk “CSP” (2017).

(^{*}) = locally finite Taylor varieties.

Theorem 4

Let $\mathbf{A}_1, \dots, \mathbf{A}_n$ be finite algebras in a locally finite WDT variety, $n \geq 3$. Suppose $R \leq_{sd} \mathbf{A}_1 \times \dots \times \mathbf{A}_n$ is completely functional and critical.

- ① Each \mathbf{A}_i is subdirectly irreducible, say with monolith μ_i .
- ② Each μ_i is abelian. Let $\alpha_i := (0 : \mu_i)$ for each $i \in [n]$.

- ③ $\mathbf{A}_1 \sim \mathbf{A}_2 \sim \dots \sim \mathbf{A}_n$, meaning

$$\mathbf{D}(\alpha_1, \mu_1) \cong \mathbf{D}(\alpha_2, \mu_2) \cong \dots \cong \mathbf{D}(\alpha_n, \mu_n).$$

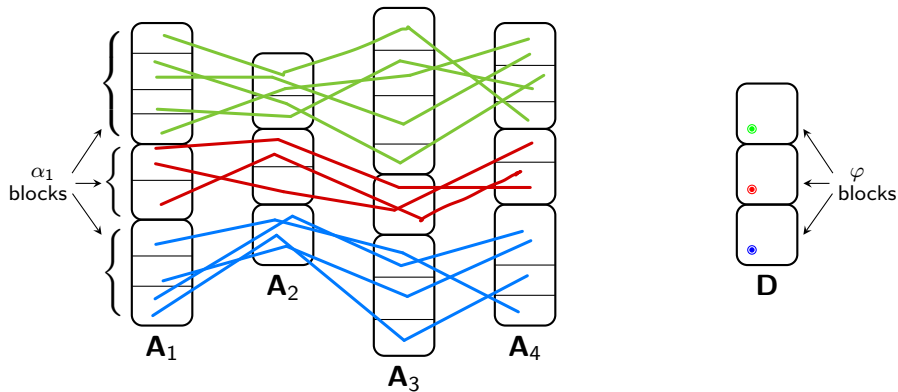
- ④ R is “annihilator-coherent,” meaning:

$$\mathbf{a}, \mathbf{b} \in R \implies \left(\forall i, j \ (a_i, b_i) \in \alpha_i \iff (a_j, b_j) \in \alpha_j \right).$$

(1) is due to Kearnes & Szendrei (and does not need “WDT” or $n \geq 3$).

(2)–(4) were proved by Kearnes & Szendrei in CM case; I extend to WDT.

Theorem 4 picture:



A completely functional, critical $R \leq_{sd} \mathbf{A}_1 \times \cdots \times \mathbf{A}_4$

Device for proving (3), (4)

To show e.g.

$$\mathbf{a}, \mathbf{b} \in R \implies \left((a_1, b_1) \in \alpha_1 \iff (a_2, b_2) \in \alpha_2 \right)$$

and $\mathbf{A}_1 \sim \mathbf{A}_2$

define $\mathbf{C} := \text{proj}_{1,2}(R) \leq_{sd} \mathbf{A}_1 \times \mathbf{A}_2$ and

$$\tau := \left\{ \begin{array}{ccc} & \xrightarrow{C} & \\ \mu_1 \downarrow & & \downarrow \mu_2 \\ & \xrightarrow{C} & \end{array} \begin{array}{c} a \quad b \\ a' \quad b' \end{array} : \exists \mathbf{x} \text{ with } (a, b, \mathbf{x}), (a', b', \mathbf{x}) \in R \right\}$$

“Zhuk Bridge”

As a binary relation on rows, $\tau \in \text{Con } \mathbf{C}$.

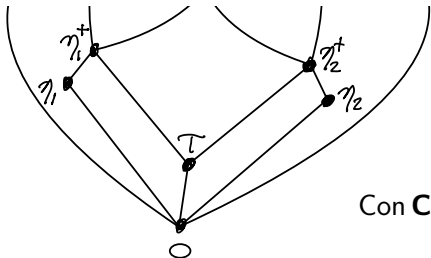
As a binary relation on columns, $\tau \leq_{sd} \mu_1 \times \mu_2$.

Using τ on rows:

$$\eta_i = \ker(\text{proj}_i) \in \text{Con } \mathbf{C}$$

$$\eta_i^+ / \eta_i \searrow \tau / 0 \text{ for } i = 1, 2.$$

So τ is abelian.



In WDT varieties, perspective abelian intervals have = relative annihilators.

So $(\eta_1 : \eta_1^+) = (0 : \tau) = (\eta_2 : \delta_2^+)$. Proves (α_1, α_2) -coherence of R .

Using τ on columns: the rule

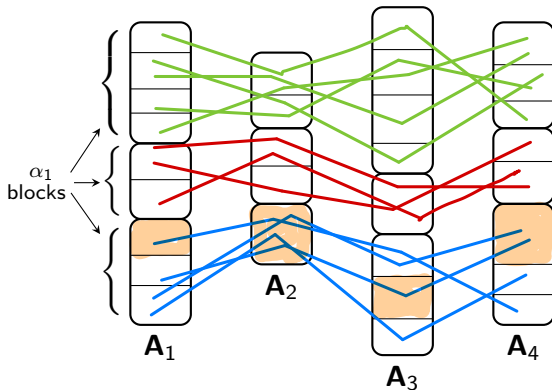
$$(a, a') / \Delta_1 \mapsto (b', b) / \Delta_2 \quad \text{for all} \quad \begin{array}{ccc} a & \xrightarrow{c} & b \\ \mu_1 \downarrow & & \downarrow \mu_2 \\ a' & \xrightarrow{c} & b' \end{array} \in \tau$$

determines a well-defined isomorphism $\lambda_{12} : \mathbf{D}(\alpha_1, \mu_1) \cong \mathbf{D}(\alpha_2, \mu_2)$.

Proves $\mathbf{A}_1 \sim \mathbf{A}_2$.

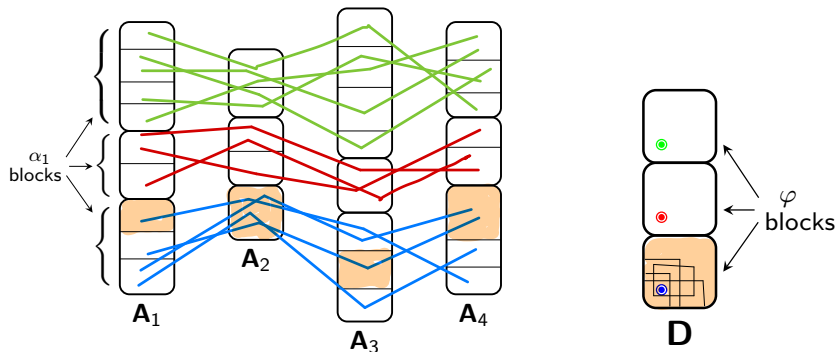
Strands

Suppose $R \leq_{sd} \mathbf{A}_1 \times \cdots \times \mathbf{A}_n$ as in Theorem 4.



A **strand** of R is a product $C = C_1 \times \cdots \times C_n$ where each C_i is a μ_i -block and $R \cap C \neq \emptyset$.

Fix a strand $C = C_1 \times \cdots \times C_n$.



Let \mathbf{D} be the common (up to \cong) difference algebra for $\mathbf{A}_1, \dots, \mathbf{A}_n$.

Let \mathbb{F} be the common finite field.

Let \mathcal{E} be the φ -block corresponding to the α_i -blocks containing C .

Embed C_1, \dots, C_n in \mathcal{E} via $\chi_{e_1}, \dots, \chi_{e_n}$ composed with \cong 's.

Intuition: $R \cap C$ “should” be the pullback to C of the solution set in \mathcal{E}^n of a single \mathbb{F} -linear equation.

I can't prove this intuition.

But I can prove it “locally” (using ideas from Zhuk “Key relations” 2017).

Let R^* be the unique upper cover of R in $\text{Sub}(\mathbf{A}_1 \times \cdots \times \mathbf{A}_n)$.

Let $C = C_1 \times \cdots \times C_n$ be a strand of R .

Define an edge relation E on $R^* \cap C$ by

$$\mathbf{a} E \mathbf{b} \iff \exists i \in [n] \text{ with } a_j = b_j \text{ for all } j \neq i.$$

Definition

An **R -block** (in C) is a connected component of the graph $(R^* \cap C, E)$.

I can verify the intuition on each R -block in C .

Blocks have a lot of structure.

Theorem 5

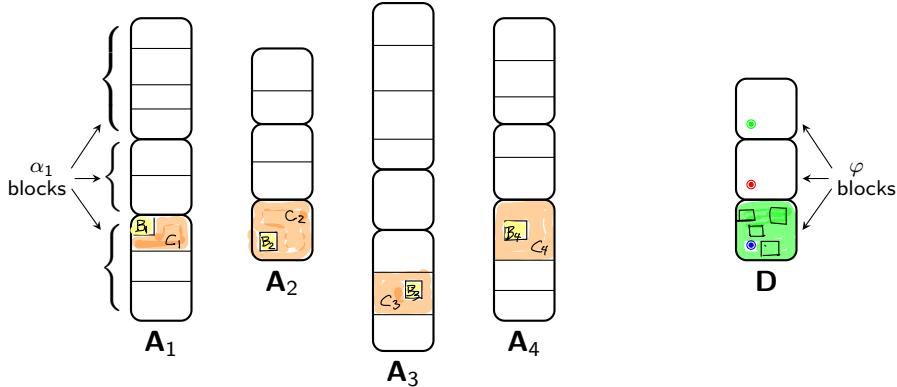
Let $R \leq_{sd} \mathbf{A}_1 \times \cdots \times \mathbf{A}_n$ as in Theorem 4.

Let $C = C_1 \times \cdots \times C_n$ be a strand of R .

Let B be an R -block in C .

- 1 $B \cap R \neq \emptyset$. (In fact, every $\mathbf{a} \in B \setminus R$ is “essentially” in $B \cap R$.)
- 2 $B = B_1 \times \cdots \times B_n$.
- 3 $|B_1| = \cdots = |B_n|$.
- 4 Each B_i is a coset of a subgroup of $\text{Grp}(\mu_i, e_i)$ ($e_i \in C_i$).
- 5 $\forall i \in [n]$, $R \cap B$ “is” the graph of a function $\prod_{j \neq i} B_j \rightarrow B_i$.

Kearnes & Szendrei proved (1).



Theorem 6

(With everything as before): If $B = B_1 \times \cdots \times B_n$ is an R -block in the strand $C = C_1 \times \cdots \times C_n$, and each C_i is embedding in the appropriate φ -block of \mathbf{D} , then $R \cap B$ is the pullback to B of the solution set of a single \mathbb{F} -linear equation

$$\lambda_1 x_1 + \cdots + \lambda_n x_n = u$$

in \mathcal{E}^n .

(Congruence modular case)

Lemma 9

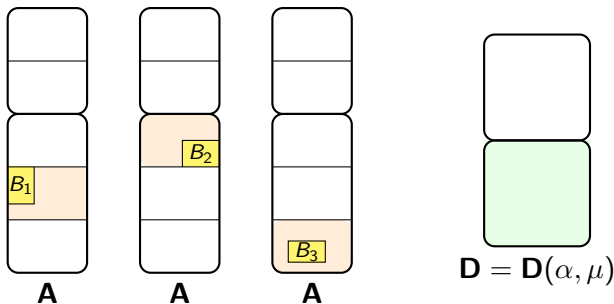
(With everything as before): If $\mathbf{A}_1, \dots, \mathbf{A}_n$ are in a congruence modular variety, then every strand of R is contained in R^* and is an R -block.

So the intuition holds in CM varieties.

Question: where do $\lambda_1, \dots, \lambda_n \in \mathbb{F}$ in Theorem 6 come from?

Consider the special case where $n = 3$ and

- $\mathbf{A}_1 = \mathbf{A}_2 = \mathbf{A}_3 =: \mathbf{A}$. (monolith μ , annihilator α)
- $(a_1, a_2, a_3) \in R \implies a_1 \overset{\alpha}{\equiv} a_2 \overset{\alpha}{\equiv} a_3$.



Recall that R induces isomorphisms $\lambda_{ij} : \mathbf{D}(\alpha_i, \mu_i) \cong \mathbf{D}(\alpha_j, \mu_j)$.

I.e., $\lambda_{ij} \in \text{Aut}(\mathbf{D})$. Can check that $\lambda_{ij}(\varphi) \subseteq \varphi$ and $\lambda_{ij}(0_E) = 0_E \ \forall E$.

The linear equations used to define $R \cap B$, for each R -block B , use the same coefficients $\lambda_1, \dots, \lambda_n$. Only the right-hand side constants vary.

Question 1: what (if anything) can be said about the right-hand side constants, for R -blocks within a fixed strand C ?

If $B = B_1 \times \dots \times B_n$ is an R -block, then B_1, \dots, B_n cosets of subgroups of their respective μ_i -blocks.

Question 2: Are B_1, \dots, B_n cosets of \mathbb{F} -subspaces of their respective μ_i -blocks?

Thank you!

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