Abelian congruences in locally finite Taylor varieties

Tutorial – Lecture 2

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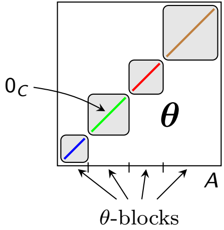
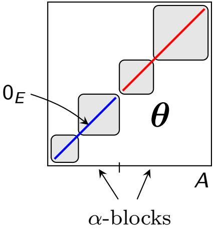
Recap – abelian congruence, weak difference term

A congruence θ of $\mathbf{A}$ is *abelian* if $\boldsymbol{\theta}$ has a congruence Δ such that each diagonal $0_C := \{(c,c) : c \in C\}$ is a Δ-block, for C a θ-block.





<u>Generalization</u>. $[\alpha, \theta] = 0$ with $\alpha \geq \theta$.   "α centralizes θ"

A 3-ary term $d(x,y,z)$ is a *weak difference term* (WDT) for a variety $\mathcal{V}$ if it is Maltsev on every block of an abelian congruence of any $\mathbf{A} \in \mathcal{V}$:

$$\theta \text{ abelian}, (a,b) \in \theta \implies d(a,a,b) = b = d(b,a,a).$$

Recap (continued) – abelian groups on θ-blocks

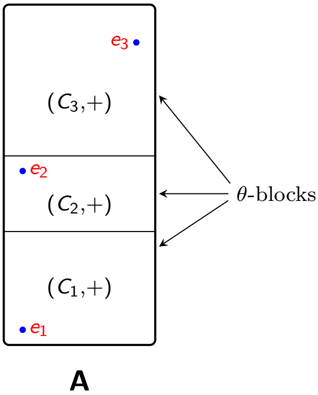
Suppose $\mathbf{A} \in \mathcal{V}$ with a WDT, $\theta \in \operatorname{Con} \mathbf{A}$, and θ is abelian.

Each θ-block C carries the structure of an abelian group $(C, +)$.

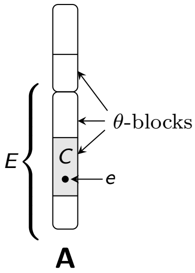
- Recipe: choose $e \in C$; then

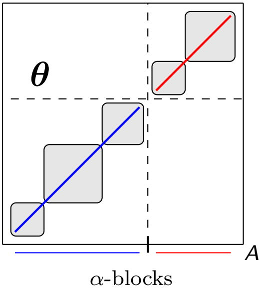
$$x + y := d(x, e, y).$$

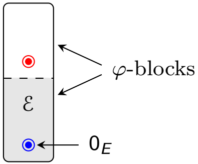
- Notation for this group: $\operatorname{Grp}(\theta, e)$.



Recap (continued) – Difference algebra $\mathbf{D}(\alpha,\theta)$







$$\mathbf{D} := \mathbf{D}(\alpha,\theta) = \boldsymbol{\theta}/\Delta$$

Suppose $\mathbf{A} \in \mathcal{V}$ with a WDT, $\alpha \geq \theta$ in $\operatorname{Con}\mathbf{A}$, $[\alpha,\theta] = 0$, and $\Delta = \Delta_{\theta,\alpha}$ the smallest witness. Let $\mathbf{D} = \mathbf{D}(\alpha,\theta) := \boldsymbol{\theta}/\Delta$ and $\varphi := \overline{\alpha}/\Delta$.

1. θ and φ are abelian. (So lots of abelian groups.)
2. If E is an α-block and $\mathcal{E}$ is the corresponding φ-block, then the maps $\boxed{\chi_e : a \mapsto (a,e)/\Delta}$ embed the various groups $\operatorname{Grp}(\theta,e)$ $(e \in E)$ into the single group $\operatorname{Grp}(\varphi, 0_E)$.

Today's goal: examine the abelian groups

$$\mathrm{Grp}(\theta, e)$$

when **A** is finite and θ is minimal, i.e., $0 \prec \theta$.

We will see that the groups become vector spaces over a finite field $\mathbb{F}$.

Freese (1983) proved this for **A** in congruence modular varieties.

- That it also holds for **A** in WDT varieties is folklore.
- I give two proofs for WDT varieties in my "Similarity" manuscript.
- Today I sketch a 3rd (even better) proof.

Constructing the finite field $\mathbb{F}$

(when $\mathbf{A}$ is finite and $0 \prec \theta$)

Easy case: when θ has a transversal $T \leq \mathbf{A}$

Until further notice, assume:

- $\mathbf{A} \in \mathcal{V}$ with a WDT, with $\mathbf{A}$ finite
- $\theta \in \operatorname{Con} \mathbf{A}$, θ is abelian, and $0 \prec \theta$.
- T is a transversal for θ satisfying $T \leq \mathbf{A}$.

Definition

$$F := \{\lambda \in \operatorname{End}(\mathbf{A}) \,:\, \lambda(\theta) \subseteq \theta \text{ and } \lambda(e) = e \text{ for all } e \in T\}.$$

Easy observation:

- $\lambda \in F \implies \forall\, \theta\text{-block } C,\ \ \lambda|_C \in \operatorname{End}(\operatorname{Grp}(\theta, e))$ where $e \in C \cap T$.

$F := \{\lambda \in \mathrm{End}(\mathbf{A}) : \lambda(\theta) \subseteq \theta \text{ and } \lambda(e) = e \text{ for all } e \in T\}$

Given $\lambda, \mu \in F$, define $\lambda + \mu : A \to A$ so that for every θ-block C,

$$(\lambda + \mu)|_C = \lambda|_C + \mu|_C \ \text{ in } \mathrm{End}(\mathrm{Grp}(\theta, e)) \ \text{ where } e \in C \cap T.$$

I.e.,

$$(\lambda + \mu)(a) = d(\lambda(a), e, \mu(a)) \quad \text{where } a \overset{\theta}{\equiv} e \in T.$$

Lemma 5

$\lambda, \mu \in F \implies \lambda + \mu \in F.$

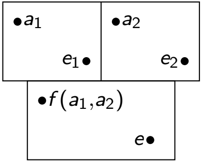
Proof sketch. Nontrivial part: $\lambda, \mu \in F \implies \lambda + \mu \in \mathrm{End}(\mathbf{A})$.

Let $f(x, y)$ be a basic operation.

Let $a_1, a_2 \in A$. Compute $f(a_1, a_2)$.

Let $e_1, e_2, e \in T$ in their θ-blocks.

$T \leq \mathbf{A} \implies \boxed{f(e_1, e_2) = e.}$



θ-blocks

$$e_1, e_2, e \in T, \quad a_i \overset{\theta}{\equiv} e_i, \quad f(a_1, a_2) \overset{\theta}{\equiv} f(e_1, e_2) = e$$

Now let $\lambda, \mu \in F$.

Let $\sigma := \lambda + \mu$. Then:

$$f(\sigma(a_1), \sigma(a_2)) = f\left(d(\underbrace{\lambda(a_1), e_1, \mu(a_1)}_{\theta}),\ d(\underbrace{\lambda(a_2), e_2, \mu(a_2)}_{\theta})\right)$$

$$= d\left(f(\lambda(a_1), \lambda(a_2)),\ f(e_1, e_2),\ f(\mu(a_1), \mu(a_2))\right)$$

Technical Lemma 1

$$= d\left(\lambda(f(a_1, a_2)),\ e,\ \mu(f(a_1, a_2)\right) \qquad \lambda, \mu \in \mathrm{End}(\mathbf{A})$$

$$= \sigma(f(a_1, a_2)).$$

Similarly for basic operations of other arities.

Thus $\sigma \in \mathrm{End}(\mathbf{A})$. □

Corollary 2

(In the easy case: θ has a transversal $T \leq \mathbf{A}$)

$\mathbb{F} := (F; +, \circ, 0, 1)$ is a unital ring, where

$0 : A \to A$ is the retraction of $\mathbf{A}$ onto $\mathbf{T}$ collapsing θ-blocks

$1 : A \to A$ is the identity map.

$\mathbb{F}$ acts naturally on each θ-block $\mathrm{Grp}(\theta, e)$ ($e \in T$), turning each into an $\mathbb{F}$-module.

We haven't yet used finiteness or $0 \prec \theta$.

We use them now to prove $\mathbb{F}$ is a finite field.

Key Lemma 6

(Easy case, $0 \prec \theta$): T is a maximal proper subuniverse of $\mathbf{A}$.

Proof sketch (if time). Suppose $T < S \leq \mathbf{A}$.

Pick $a \in S \setminus T$. Let $b \in A$. (Must show $b \in S$.)

Let $e, e' \in T$ with $a \overset{\theta}{\equiv} e$ and $b \overset{\theta}{\equiv} e'$. Note: $a \neq e$ (since $a \notin T$).

θ minimal $\implies$ $(b, e') \in \mathrm{Cg}(a, e)$.

θ abelian $\implies$ $\exists\, g \in \mathrm{Pol}_1(\mathbf{A})$ with $\boxed{g(a) = b \ \text{ and } \ g(e) = e'}$ (Lemma 1).

Write $g(x) = t(x, c_1, \ldots, c_n)$ with $t(x, \mathbf{y})$ a term.

Choose $e_1, \ldots, e_n \in T$ with $c_i \overset{\theta}{\equiv} e_i$. Observe:

$$\underbrace{t(e, e_1, \ldots, e_n)}_{\in T} \overset{\theta}{\equiv} t(e, c_1, \ldots, c_n) = g(e) = e'.$$

So

$$t(e, e_1, \ldots, e_n) = e' = t(e, c_1, \ldots, c_n).$$

(Proof continued next page)

$T < S \leq A, \quad a \in S \setminus T, \quad b \in A. \quad$ (Must show $b \in S$.)

$e_1 \overset{\theta}{\equiv} c_1, \ldots, e_n \overset{\theta}{\equiv} c_n, \quad e \overset{\theta}{\equiv} a,$ and $e, e_1, \ldots, e_n \in T$

$t(e, e_1, \ldots, e_n) = e' = t(e, c_1, \ldots, c_n) \qquad\qquad b = t(a, c_1, \ldots, c_n)$

Let $\Delta \in \operatorname{Con} \boldsymbol{\theta}$ witness that θ is abelian. Observe that

$$t\left(\begin{pmatrix} e \\ e \end{pmatrix}, \begin{pmatrix} e_1 \\ c_1 \end{pmatrix}, \ldots, \begin{pmatrix} e_n \\ c_n \end{pmatrix}\right) \overset{\Delta}{\equiv} t\left(\begin{pmatrix} a \\ a \end{pmatrix}, \begin{pmatrix} e_1 \\ c_1 \end{pmatrix}, \ldots, \begin{pmatrix} e_n \\ c_n \end{pmatrix}\right)$$

Δ

This simplifies to

$$\begin{pmatrix} e' \\ e' \end{pmatrix} \overset{\Delta}{\equiv} \begin{pmatrix} t(a, e_1, \ldots, e_n) \\ b \end{pmatrix}.$$

$\Longrightarrow \; t(a, e_1, \ldots, e_n) = b$

$\Longrightarrow \; b \in \operatorname{Sg}(\{a\} \cup T) \subseteq S.$

(Completes the proof that T is a maximal proper subuniverse of $\mathbf{A}$.) $\square$

Suppose $\lambda \in \mathbb{F}$, $\lambda \neq 0$.

Observe: $T \leq \mathrm{ran}(\lambda) \leq \mathbf{A}$ and $T \leq \lambda^{-1}(T) \leq \mathbf{A}$.

$\lambda \neq 0 \implies \mathrm{ran}(\lambda) \neq T$ and $\lambda^{-1}(T) \neq A$.

Hence (1) $\boxed{\mathrm{ran}(\lambda) = A}$ and (2) $\boxed{\lambda^{-1}(T) = T}$.

(1) $\implies \lambda$ is surjective.

(2) $\implies$ each $\lambda|_C$ is injective $\implies \lambda$ is injective

$\implies \lambda^{-1}$ exists; so $\mathbb{F}$ is a division ring. Thus $\mathbb{F}$ is a field (by finiteness).

Theorem 2

If $\mathbf{A}$ is finite in a WDT variety, θ is abelian, $0 \prec \theta$, and T is a transversal of θ $\boxed{\text{satisfying } T \leq \mathbf{A}}$, then $\mathbb{F}$ constructed above is a finite field, and for each $e \in T$, $\mathrm{Grp}(\theta, e)$ is naturally a vector space over $\mathbb{F}$, $\forall e \in T$.

Problem:

A transversal T for θ satisfying $T \leq \mathbf{A}$ might not exist.

- For example, the quaternion group $\mathbf{Q}$ and $\theta = \{(x, y) : y = \pm x\}$.

Solution:

Choose $\alpha \geq \theta$ satisfying $[\alpha, \theta] = 0$. (E.g., $\alpha = \theta$)

Form $\mathbf{D} = \mathbf{D}(\alpha, \theta)$ and its derived congruence φ.

$0 \prec \theta \implies 0 \prec \varphi$ (can show).

And φ has a natural transversal T_α satisfying $T_\alpha \leq \mathbf{D}$, namely,

$$T_\alpha = \{0_E : E \text{ is an } \alpha\text{-block}\}.$$

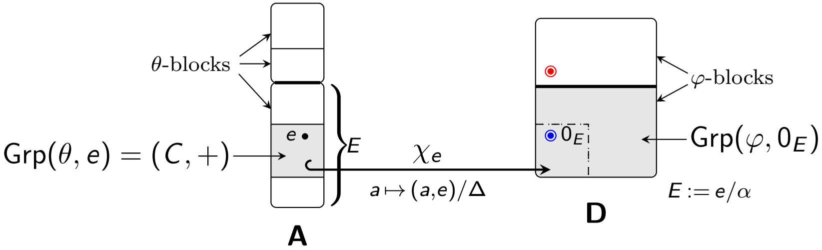
So we can define $\mathbb{F} = \mathbb{F}_{(\mathbf{D}, \varphi, T_\alpha)}$ as in the easy case, using $\mathbf{D}, \varphi, T_\alpha$.

Problem:

But $\mathbb{F}_{(\mathbf{D},\varphi,T_\alpha)}$ acts on φ-blocks (in **D**), not on θ-blocks (in **A**).

Solution.

Given $e \in A$, consider $\chi_e : \mathrm{Grp}(\theta, e) \longrightarrow \mathrm{Grp}(\varphi, 0_E)$ where $E := e/\alpha$.



If we show $\mathrm{ran}(\chi_e)$ is an $\mathbb{F}_{(\mathbf{D},\varphi,T_\alpha)}$-subspace of $\mathrm{Grp}(\varphi, 0_E)$ ...

... then we can define an action of $\mathbb{F}_{(\mathbf{D},\varphi,T_\alpha)}$ on C via χ_e and χ_e^{-1}.

Lemma 7

Suppose **A** is finite, $\mathbf{A} \in \mathcal{V}$ with WDT, $0 \prec \theta \leq \alpha$, and $[\alpha, \theta] = 0$.

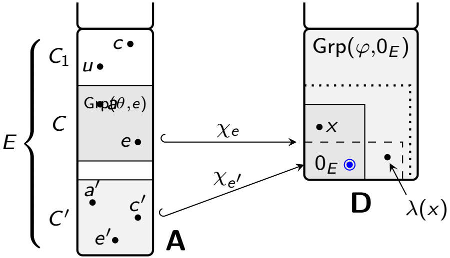
Let $\mathbf{D} := \mathbf{D}(\alpha, \theta)$, φ its derived congruence, and $\mathbb{F} := \mathbb{F}_{(\mathbf{D}, \varphi, T_\alpha)}$.

If $e \in A$ and $E = e/\alpha$, then $\operatorname{ran}(\chi_e)$ is an $\mathbb{F}$-subspace of $\operatorname{Grp}(\varphi, 0_E)$.

Proof. Fix $\lambda \in \mathbb{F}$, let $C := e/\theta$. Must show $\operatorname{ran}(C)$ is closed under λ.

Let $x \in \operatorname{ran}(C)$, so $x = (a, e)/\Delta$ with $a \in C$. (Can assume $a \neq e$.)

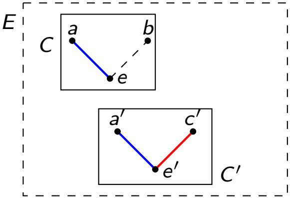
Write $\lambda(x) = (c, u)/\Delta$ and $C_1 := u/\theta$.



Trick: $\exists$ θ-block $C' \subseteq E$ with

$\operatorname{ran}(C) \cup \operatorname{ran}(C_1) \subseteq \operatorname{ran}(C')$.

Get $a', c', e' \in C'$ with
$(a', e')/\Delta = x$
$(c', e')/\Delta = \lambda(x)$.



$$x = (a, e)/\Delta = (a', e')/\Delta$$
$$\lambda(x) = (c', e')/\Delta$$

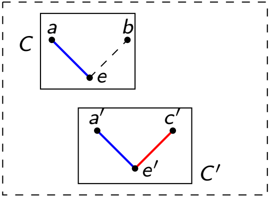
Recall: $a \neq e$. So $a' \neq e'$.

$$\theta \text{ minimal } \implies (a, e) \in \mathrm{Cg}(a', e')$$
$$\implies \exists f \in \mathrm{Pol}_1(\mathbf{A}) \text{ s.t. } f(a') = a \text{ and } f(e') = e. \qquad \text{(Lemma 1)}$$

Observe that $f(C') \subseteq C$.

Define $b = f(c')$.

Claim: $(b, e)/\Delta = (c', e')/\Delta \ \ (= \lambda(x))$.



$$x = (a,e)/\Delta = (a',e')/\Delta \in \mathbf{D}$$
$$\lambda(x) = (c',e')/\Delta \in \mathbf{D}$$
$$f \in \mathrm{Pol}_1(\mathbf{A}), \quad f(e') = e, \ f(a') = a, \ f(c') = b$$

Write $f(x) = t(x, c_1, \ldots, c_n)$, $E_i = c_i/\alpha$

$$\begin{aligned} t(x, 0_{E_1}, \ldots, 0_{E_n}) &= t((a',e')/\Delta,\ (c_1,c_1)/\Delta, \ldots, (c_n,c_n)/\Delta) \\ &= (f(a'), f(e'))/\Delta \\ &= (a,e)/\Delta = x \end{aligned}$$

so

$$\begin{aligned} \lambda(x) &= \lambda(t(x, 0_{E_1}, \ldots, 0_{E_n})) \\ &= t(\lambda(x), 0_{E_1}, \ldots, 0_{E_n}) && \lambda \in F_{(\mathbf{D},\varphi,T_\alpha)} \\ &= t((c',e')/\Delta,\ (c_1,c_1)/\Delta, \ldots, (c_n,c_n)/\Delta) \\ &= (f(c'), f(e'))/\Delta \\ &= (b,e)/\Delta, \quad \text{proving the Claim.} && \text{Hence } \lambda(x) \in \mathrm{ran}(C). \end{aligned}$$

Theorem 3

Suppose **A** is finite in a WDT variety, $0 \prec \theta \leq \alpha$, and $[\alpha, \theta] = 0$.

Let $\mathbf{D} = \mathbf{D}(\alpha, \theta)$, φ its derived congruence, and $\mathbb{F} := \mathbb{F}_{(\mathbf{D}, \varphi, T_\alpha)}$.

1. For every $e \in A$, $\mathbb{F}$ acts naturally on $\mathrm{Grp}(\theta, e)$, turning it into a vector space over $\mathbb{F}$.
2. The action is this: letting $C = e/\theta$, then for every $\lambda \in \mathbb{F}$ and $a \in C$,

$$\lambda \cdot a := \text{the unique } b \in C \text{ satisfying } (b, e)/\Delta = \lambda((a, e)/\Delta).$$

3. With respect to these actions, the maps $\chi_e : \mathrm{Grp}(\theta, e) \longhookrightarrow \mathrm{Grp}(\varphi, 0_E)$ become $\mathbb{F}$-linear embeddings of vector spaces.

Discussion

<u>Problem</u>:

Why didn't I just let $\alpha = \theta$ and use $\mathbf{D} = \mathbf{D}(\theta, \theta)$ and $\mathbb{F} = \mathbb{F}_{(\mathbf{D}(\theta,\theta),\varphi,T_\theta)}$?

Then each α-class has only one θ-class, and we would not have needed Lemma 7 (or the definition of $[\alpha, \theta] = 0$).

<u>Answer</u>:

Come to tomorrow's lecture.

<u>Problem</u>:

Do different choices for α lead to different fields $\mathbb{F}_{(\mathbf{D},\varphi,T_\alpha)}$?

Answer:

They don't!

Proposition

Let **A** be finite in a WDT variety, and $0 \prec \theta \in \mathrm{Con}\,\mathbf{A}$ with θ abelian.

Let $\alpha \geq \theta$ satisfy $[\alpha, \theta] = 0$.

Let $\mathbb{F}_\alpha = \mathbb{F}_{(\mathbf{D}, \varphi, T_\alpha)}$ where $\mathbf{D} = \mathbf{D}(\alpha, \theta)$ and $\varphi = \overline{\alpha}/\Delta_{\theta,\alpha}$.

Then:

1. $\mathbb{F}_\alpha$ is independent of the choice of α (up to obvious isomorphisms).
2. For each $e \in A$, the actions of $\mathbb{F}_\alpha$ on $\mathrm{Grp}(\theta, e)$ are the same (modulo the obvious isomorphisms).

Summary

Suppose $\mathbf{A} \in \mathcal{V}$ with a WDT, $\mathbf{A}$ is finite, $\theta \in \operatorname{Con} \mathbf{A}$ is abelian, and $0 \prec \theta$. There is a finite field $\mathbb{F}$ such that:

1. Each θ-block naturally inherits the structure of a vector space over $\mathbb{F}$.
2. For each $\alpha \geq \theta$ with $[\alpha, \theta] = 0$:
   (a) Each φ-block of the difference algebra $\mathbf{D}(\alpha, \theta)$ inherits the structure of a vector space over $\mathbb{F}$.
   (b) Each $\chi_e : \mathrm{Grp}(\theta, e) \longrightarrow \mathrm{Grp}(\varphi, 0_E)$ is a vector space embedding.
3. (And more!) The $\mathbb{F}$-vector space structure on θ-blocks determines the restrictions of polynomials of $\mathbf{A}$ to θ-blocks, in the following sense:

   Fix a transveral T for θ.
   For all $n \geq 1$, for all θ-blocks $C_1, \ldots, C_n$, the set
   $$\{p|_{C_1 \times \cdots \times C_n} : p \in \mathrm{Pol}_n(\mathbf{A})\}$$
   is exactly the set of all $\mathbb{F}$-affine maps
   $$\mathrm{Grp}(\theta, e_1) \times \cdots \times \mathrm{Grp}(\theta, e_n) \to \mathrm{Grp}(\theta, e)$$
   where $e_i \in C_i \cap T$ and $e \in T$.