Abelian congruences in locally finite Taylor varieties

Tutorial – Lecture 1

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What is this tutorial about?

Shameless flogging of my recent papers:

1. "Abelian congruences and similarity in varieties with a weak difference term" (arXiv 2025)
2. "Zhuk's bridges, centralizers, and similarity" (arXiv 2025)
3. "Critical rectangular relations in locally finite Taylor varieties" (coming).

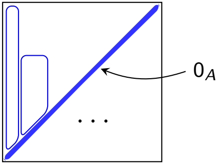
Plan

1. (Today)
   - Abelian congruences, weak difference terms
   - Centrality, difference algebras
   - Embedding congruence blocks, ranges
2. (Wednesday)
   - The finite field associated to an abelian minimal congruence of a finite Taylor algebra
3. (Thursday)
   - Critical, completely functional relations in locally finite Taylor varieties

Part 1 – abelian congruences, weak difference terms

Definition

An algebra $\mathbf{A}$ is *abelian* if $\mathbf{A}^2$ has a congruence Δ for which the diagonal $0_A := \{(a,a) : a \in A\}$ is a single block.

$\mathbf{A}^2 =$ 

Example: An abelian group $\mathbf{A} = (A, +)$ is abelian.

Proof: $0_A \lhd \mathbf{A}^2$. So 0_A is a block of the congruence Δ of $\mathbf{A}^2$, namely

$$\begin{pmatrix} a \\ b \end{pmatrix} \overset{\Delta}{\equiv} \begin{pmatrix} a' \\ b' \end{pmatrix} \iff \begin{pmatrix} a \\ b \end{pmatrix} - \begin{pmatrix} a' \\ b' \end{pmatrix} \in 0_A$$

$$\iff a - a' = b - b'$$

$$\iff a - b = a' - b'.$$ ‘Equal differences’

Theorem 1 (Gumm, Herrmann 1979)

Suppose **A** is abelian (witnessed by Δ) and has a *Maltsev* term $m(x, y, z)$:

$$m(x, x, y) \approx y \approx m(y, x, x).$$

Fix $e \in A$, and define

$$x + y := m(x, e, y).$$

1. $+$ is an abelian group operation on A, with identity element e.
2. $m(x, y, z) = x - y + z$.
3. $\Delta = \{((a, b), (a', b')) \in A^2 \times A^2 \, : \, a - b = a' - b'\}$.
4. (And more: $+$ governs the polynomial operations of **A** . . .)

$\mathbf{A}$ is abelian. $\quad m(x,x,y) \approx y \approx m(y,x,x). \quad x+y := m(x,e,y).$

Proof of $x+y=y+x$

Let $\Delta \in \operatorname{Con} \mathbf{A}^2$ witness abelianness of $\mathbf{A}$. Let $a,b \in A$.

We have

$$\begin{pmatrix} a \\ b \end{pmatrix} \overset{\Delta}{\equiv} \begin{pmatrix} a \\ b \end{pmatrix}, \quad \begin{pmatrix} b \\ b \end{pmatrix} \overset{\Delta}{\equiv} \begin{pmatrix} e \\ e \end{pmatrix}, \quad \begin{pmatrix} b \\ a \end{pmatrix} \overset{\Delta}{\equiv} \begin{pmatrix} b \\ a \end{pmatrix}.$$

So

$$m\left(\begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} b \\ b \end{pmatrix}, \begin{pmatrix} b \\ a \end{pmatrix}\right) \overset{\Delta}{\equiv} m\left(\begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} e \\ e \end{pmatrix}, \begin{pmatrix} b \\ a \end{pmatrix}\right),$$

i.e.,

$$\begin{pmatrix} a \\ a \end{pmatrix} = \begin{pmatrix} m(a,b,b) \\ m(b,b,a) \end{pmatrix} \overset{\Delta}{\equiv} \begin{pmatrix} m(a,e,b) \\ m(b,e,a) \end{pmatrix} = \begin{pmatrix} a+b \\ b+a \end{pmatrix}.$$

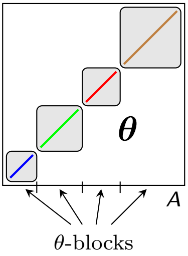
The diagonal 0_A is a Δ-block, so $a+b=b+a$.

Relativizing abelianness: to congruences

Definition

Suppose $\mathbf{A}$ is an algebra and $\theta \in \operatorname{Con} \mathbf{A}$.

1. $\boldsymbol{\theta}$:= the subalgebra of $\mathbf{A}^2$ with universe θ.
2. θ is *abelian* if $\boldsymbol{\theta}$ has a congruence Δ such that for each θ-block C, the diagonal $0_C := \{(c,c) : c \in C\}$ is a Δ-block.



Ideal situation:

- Each θ-block has an abelian group operation $+$ on it.
- $(a,b) \stackrel{\Delta}{\equiv} (a',b') \iff a,b,a',b'$ belong to the same θ-block and $a - b = a' - b'$.

"Equal differences in each group"

The theorem on abelian Maltsev algebras relativizes to congruences.

You simply need a term which satisfies Maltsev's identities **on each block of the abelian congruence.**

Corollary 1 (folklore)

Suppose **A** is an algebra, $\theta \in \operatorname{Con} \mathbf{A}$ is abelian, and **A** has a term $d(x, y, z)$ which "is Maltsev" on each θ-block.

1. For each θ-block C, if $e \in C$ and $x + y := d(x, e, y)$, then $(C, +)$ is an abelian group.
2. The smallest $\Delta \in \operatorname{Con} \boldsymbol{\theta}$ witnessing abelianness of θ is the "equal differences in each group" relation.
3. (And more: the operations $+$ on the θ-blocks govern the restrictions of polynomials to tuples of θ-blocks . . .)

Definition (Kearnes 1995, Lipparini 1996)

A term $d(x, y, z)$ which is Maltsev on each block of every abelian congruence (of every algebra in a variety) is called a **weak difference term** (or **WDT**) for the variety.

Nearly all varieties of interest have a weak difference term, including:

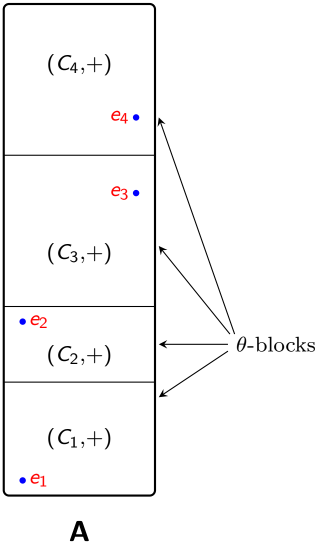
- congruence modular varieties (Gumm 1980)
- <u>locally finite Taylor varieties</u> (Hobby & McKenzie 1988)

Recent papers with a focus on varieties with a WDT:

- Kearnes & Kiss, *The Shape of Congruence Lattices*, 2013
- Kearnes, Relative Maltsev definability. . . , 2023
- Kearnes & Kiss, What is the weakest idempotent. . . (arXiv 2024)

The picture

Blocks of an abelian congruence θ
(in a WDT variety):



Notation: $\boxed{\mathrm{Grp}(\theta, e)}$ denotes $(C, +)$

where $C = e/\theta$ and $x + y := d(x, e, y)$.

1980s notation: $M(\theta, e)$

Useful facts for the experts:

Technical Lemma 1

Suppose **A** belongs to a variety with a WDT, $\theta \in \operatorname{Con} \mathbf{A}$, and θ is abelian.

1. For all $\delta \in \operatorname{Con} \mathbf{A}$, $\theta \vee \delta = \delta \circ \theta \circ \delta$.
2. For all $\delta \in \operatorname{Con} \mathbf{A}$, $(\theta \vee \delta)/\delta$ is abelian.
3. $(a, b) \in \theta \implies \operatorname{Cg}(a, b) = \{(f(a), f(b)) : f \in \operatorname{Pol}_1(\mathbf{A})\}$.
4. For all $f \in \operatorname{Pol}_k(\mathbf{A})$, for all $a_i \stackrel{\theta}{\equiv} b_i \stackrel{\theta}{\equiv} c_i$ $(1 \leq i \leq k)$,
$$f(d(a_1, b_1, c_1), \ldots, d(a_k, b_k, c_k)) = d(f(\mathbf{a}), f(\mathbf{b}), f(\mathbf{c})).$$

The proofs are elementary. For example (if time):

(1) Show $\theta \circ \delta \circ \theta \subseteq \delta \circ \theta \circ \delta$. Assume $a \stackrel{\theta}{\equiv} x \stackrel{\delta}{\equiv} y \stackrel{\theta}{\equiv} b$. Then

$$a = d(a, x, x) \stackrel{\delta}{\equiv} d(a, y, y) \stackrel{\theta}{\equiv} d(x, y, b) \stackrel{\delta}{\equiv} d(y, y, b) = b.$$

(2) $\boxed{\theta \text{ abelian} \implies (\theta \vee \delta)/\delta \text{ abelian}}$

Let Δ be the smallest witness to abelianness of θ.

Using (1), it is enough to assume $\delta \leq \theta$ and show the following:

$$\begin{pmatrix} a \\ b \end{pmatrix} \overset{\Delta}{\equiv} \begin{pmatrix} a' \\ b' \end{pmatrix} \text{ and } (a,b) \in \delta \implies (a',b') \in \delta. \qquad (*)$$

Assume the hypotheses of $(*)$. Then by the folklore Corollary,

a, b, a', b' are in a common θ-block C and

$$a - b = a' - b' \quad \text{in } (C, +)$$

so

$$\begin{aligned} b' &= a' - a + b \\ &= d(a', a, b) \end{aligned}$$

so

$$b' = d(a', a, b) \overset{\delta}{\equiv} d(a', a, a) = a'$$

as required. □

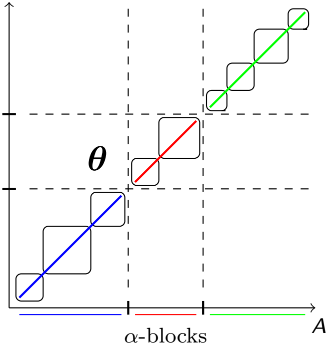
Part 2 – centrality, difference algebras

Let $\mathbf{A}$ be an algebra and $\alpha, \theta \in \operatorname{Con} \mathbf{A}$.

Definition

We say that α **centralizes** θ, and write $[\alpha, \theta] = 0$, if $\boldsymbol{\theta}$ has a congruence Δ such that for each $\underline{\alpha\text{-block } E}$, the diagonal 0_E is a Δ-block.

Picture when $\alpha \geq \theta$.

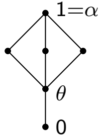


Example

Let $\mathbf{A}$ = the quaternian group $\mathbf{Q} = (\{\pm 1, \pm i, \pm j, \pm k\}, \cdot)$

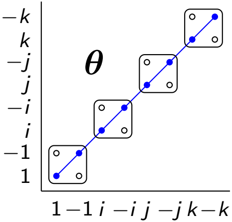
θ = the congruence corresp. to $\{\pm 1\} \lhd \mathbf{Q}$

$\alpha = 1 \quad (= Q^2)$.



Con **Q**

So $\boldsymbol{\theta} = \{(x,y) \in Q^2 : y = \pm x\} \leq \mathbf{Q}^2$ (a subgroup of order 16)

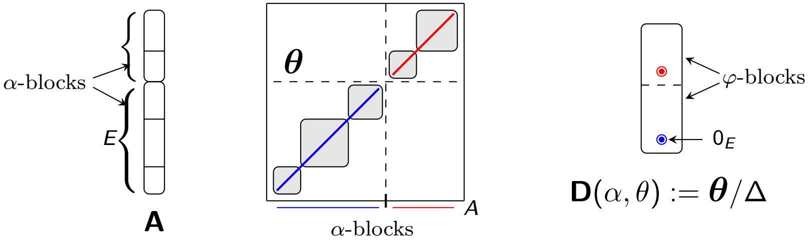


Observe that $0_Q = \{(x,x) : x \in Q\} \lhd \boldsymbol{\theta}$.

So 0_Q is a block of a congruence Δ of $\boldsymbol{\theta}$. Δ witnesses $[1,\theta] = 0$.

Construction

Assume $\alpha \geq \theta$ and $[\alpha, \theta] = 0$. Let $\Delta =$ the smallest witness. (*)



The **difference algebra** for (α, θ) is $\mathbf{D}(\alpha, \theta) := \boldsymbol{\theta}/\Delta$.

Let $\overline{\alpha} :=$ the congruence of $\boldsymbol{\theta}$ corresponding to α. Then $\Delta \leq \overline{\alpha}$.

Define

$$\varphi := \overline{\alpha}/\Delta \in \mathrm{Con}\,\mathbf{D}(\alpha, \theta), \quad \text{the } \textbf{derived congruence}.$$

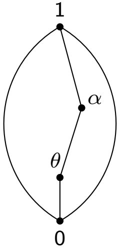
(*) $\Delta = \Delta_{\theta,\alpha} =$ the transitive closure of $\left\{ \left\langle \begin{pmatrix} r \\ s \end{pmatrix}, \begin{pmatrix} r' \\ s' \end{pmatrix} \right\rangle : \begin{pmatrix} r & r' \\ s & s' \end{pmatrix} \text{ is a } \theta,\alpha\text{-matrix} \right\}$.

Lemma 2

In varieties with a WDT, the derived congruence $\varphi = \overline{\alpha}/\Delta$ is abelian.

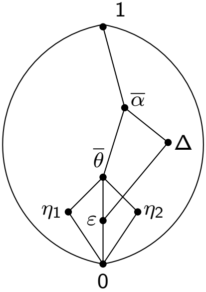
Proof sketch.

Con $\mathbf{A}$ =



Con $\boldsymbol{\theta}$ =

η_i:=ker(proj$_i$)



Let $\overline{\theta}$ = the congruence of $\boldsymbol{\theta}$ corresponding to θ.

θ is abelian $\implies$ $\overline{\theta}$ is abelian

By Technical Lemma 1, $(\overline{\theta} \vee \Delta)/\Delta$ is abelian.

Claim: $\overline{\theta} \vee \Delta = \overline{\alpha}$.

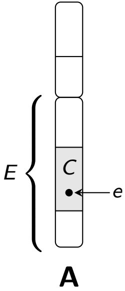
Proof: $(a, b) \overset{\overline{\alpha}}{\equiv} (a', b') \implies \begin{pmatrix} a \\ b \end{pmatrix} \overset{\overline{\theta}}{\equiv} \begin{pmatrix} a \\ a \end{pmatrix} \overset{\Delta}{\equiv} \begin{pmatrix} a' \\ a' \end{pmatrix} \overset{\overline{\theta}}{\equiv} \begin{pmatrix} a' \\ b' \end{pmatrix}$.

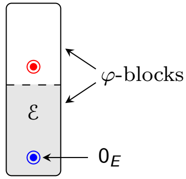
Part 3 – Embedding θ-blocks, ranges

Embeddings

Let $\mathbf{A} \in$ WDT variety, $\alpha \geq \theta$, $[\alpha, \theta] = 0$.

Let $\mathbf{D} = \mathbf{D}(\alpha, \theta)$ and $\varphi = \overline{\alpha}/\Delta$.





Abelian groups!!

Fix $e \in A$, let $C = e/\theta$.

$\mathrm{Grp}(\theta, e) := (C, +)$.

Fix $E \in A/\alpha$, let $\mathcal{E} = 0_E/\varphi$.

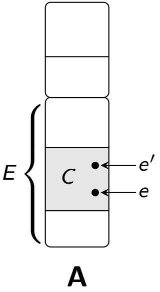
$\mathrm{Grp}(\varphi, 0_E) := (\mathcal{E}, +)$.

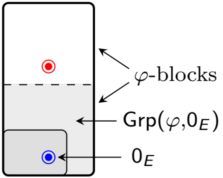
If $C \subseteq E$, define $\boxed{\chi_e : C \to \mathcal{E} \text{ by } \chi_e(a) := (a, e)/\Delta.}$

Lemma 3

χ_e is a group embedding $\mathrm{Grp}(\theta, e) \longrightarrow \mathrm{Grp}(\varphi, 0_E)$.

Fact: If C is a θ-block and $e, e' \in C$, then $\text{ran}(\chi_e) = \text{ran}(\chi_{e'})$.





$\mathbf{D} := \mathbf{D}(\alpha, \theta)$

$\boxed{\text{ran}(C)} := \text{ran}(\chi_e)$ for any $e \in C$.

Lemma 4

Fix an α-block E.

For all θ-blocks $C_1, C_2 \subseteq E$, there exists a θ-block $C \subseteq E$ such that

$$\text{ran}(C_1) \cup \text{ran}(C_2) \subseteq \text{ran}(C).$$

Proof hint: Let $C = d(C_1, C_1, C_2)$ in $\mathbf{A}/\theta$.

Proof sketch (if time). (E an α-block; C_1, C_2 two θ-blocks $\subseteq E$)

Fix $e_1 \in C_1$ and $e_2 \in C_2$. So $(e_1, e_2) \in \alpha$.

Let $e = d(e_1, e_1, e_2)$ and $C = e/\theta$.

$$e \stackrel{\alpha}{\equiv} d(e_1, e_1, e_1) = e_1 \implies e \in E \implies C \subseteq E.$$

Claim. $\mathrm{ran}(C_1) \subseteq \mathrm{ran}(C)$.

Proof. Fix $a \in C_1$. (We want $\chi_{e_1}(a) \in \mathrm{ran}(\chi_e) = \mathrm{ran}(C)$.)

Let $b := d(a, e_1, e_2) \stackrel{\theta}{\equiv} d(e_1, e_1, e_2) = e$. (So $b \in C$)

$$d\left(\begin{pmatrix} a \\ e_1 \end{pmatrix}, \begin{pmatrix} e_1 \\ e_1 \end{pmatrix}, \begin{pmatrix} e_1 \\ e_1 \end{pmatrix}\right) \stackrel{\Delta}{\equiv} d\left(\begin{pmatrix} a \\ e_1 \end{pmatrix}, \begin{pmatrix} e_1 \\ e_1 \end{pmatrix}, \begin{pmatrix} e_2 \\ e_2 \end{pmatrix}\right)$$

Δ

i.e.,

$$(a, e_1) \stackrel{\Delta}{\equiv} (b, e) \quad \text{so} \quad \chi_{e_1}(a) = \chi_e(b) \in \mathrm{ran}(C).$$

A similar argument shows $\mathrm{ran}(C_2) \subseteq \mathrm{ran}(C)$. □

Summary

Suppose $\mathbf{A} \in \mathcal{V}$ with a WDT, $\alpha \geq \theta$ in Con $\mathbf{A}$, and $[\alpha, \theta] = 0$.

Let $\Delta =$ the smallest witness, $\mathbf{D} = \boldsymbol{\theta}/\Delta$, and $\varphi = \overline{\alpha}/\Delta \in$ Con $\mathbf{D}$.

1. φ (like θ) is abelian.
2. The φ-blocks in $\mathbf{D}$ (like the θ-blocks in $\mathbf{A}$) support abelian groups.
3. The φ-blocks in $\mathbf{D}$ are naturally in 1-1 correspondence with the α-blocks in $\mathbf{A}$: $E \mapsto 0_E/\varphi$.
4. Abelian groups on θ-blocks within a fixed α-block E naturally embed into the corresponding group $\mathrm{Grp}(\varphi, 0_E)$.
5. The ranges in $\mathrm{Grp}(\varphi, 0_E)$ of the θ-blocks in E form a directed set of subgroups of $\mathrm{Grp}(\varphi, 0_E)$.