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Lattice-theoretic principles for synthetic higher category theory

Jonathan Weinberger

jww Daniel Gratzer & Ulrik Buchholtz and Nikolai Kudasov & Emily Riehl

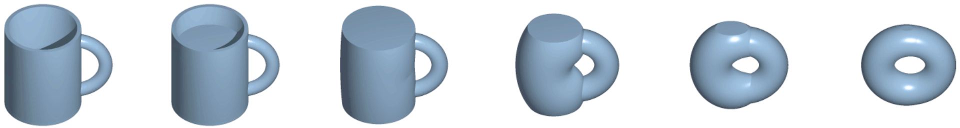
CHAPMAN UNIVERSITY

The Fletcher Jones Foundation

BLAST 2025
University of Colorado, Boulder, CO, May 23, 2025

A shift in foundations

Mathematics in the 21st century is fundamentally concerned with *deformations* $\rightsquigarrow$ Homotopy theory is primary



Animated image created by: Lucas Vieira (LucasVB), `https://en.wikipedia.org/wiki/File:Mug_and_Torus_morph.gif`

*Instead of **sets**, clouds of discrete elements, we envisage some sorts of vague **spaces**, which can be very severely deformed, mapped one to another, and all the while the specific space is not important, but only the space **up to deformation**. I am pretty strongly convinced that there is an **ongoing reversal** in the collective consciousness of mathematicians: the [. . .] **homotopical picture** of the world becomes the **basic intuition***

—Y. Manin ’09 (emphases mine)

Homotopy type theory

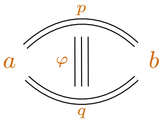
- **Homotopy type theory (HoTT):** basic types are *homotopy types*
- Reinterpretation & extension of constructive **Martin-Löf type theory** (1970s) into ∞-groupoids (even all ∞-toposes)
- New and rapidly developing field: Hofmann–Streicher ('98, '06), Voevodsky ('06/'09), Awodey–Warren ('07), van den Berg–Garner ('08, '10), Joyal, Kapulkin–Lumsdaine ('12), Shulman ('12, '19), Cisinski ('14), Coquand ('14/'15), . . .
- Deep and far-reaching bridge between **homotopy theory** and **logic/formalization**!
- Develop homotopy theory **synthetically** rather than **analytically**
- Voevodsky's **Univalence Axiom:** *homotopy equivalent types are equal*! $\rightsquigarrow$ **Univalent Foundations (UF)**

MATHEMATICS

The Origins and Motivations of Univalent Foundations

A Personal Mission to Develop Computer Proof Verification to Avoid Mathematical Mistakes

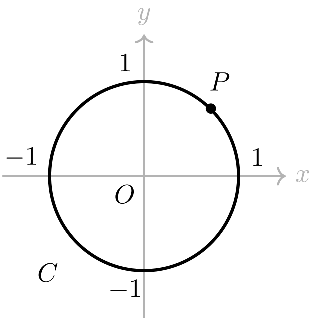
Vladimir Voevodsky • Published 2014



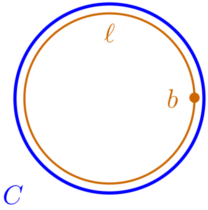
Identity types

$$a, b : A$$
$$p, q : (a =_A b)$$
$$\varphi : (p =_{(a =_A b)} q)$$
$$\dots$$

Example: the unit circle



the set of points
$C = \{P \mid \mathrm{dist}(O, P) = 1\}$



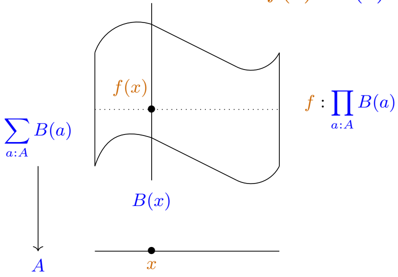
the type C generated by:

$\begin{cases} \text{a point} & b : C \\ \text{a loop} & \ell : b =_C b \end{cases}$

Dependent type formers

- Terms of **product type**: pairs $\langle a, b \rangle : A \times B$ with $a : A$ and $b : B$.
- **Dependent generalization**: A type and $a : A \vdash B(a)$ dep. type $\rightsquigarrow$ **dep. pair** or **dep. *sum* type** $\sum_{a:A} B(a)$ whose elements are pairs $\langle a, b \rangle$ with $a : A$ and $b : B(a)$.

- Terms of **function type**: functions $f : A \to B$ taking $a : A$ to $f(a) : B$.
- **Dependent generalization**: A type and $a : A \vdash B(a)$ dep. type $\rightsquigarrow$ **dep. function** or **dep. *product* type** $\prod_{a:A} B(a)$ whose elements are functions (**sections**) f taking $a : A$ to $f(a) : B(a)$



Identity type: Formation and introduction

- Per Martin-Löf's **identity types** from the 1970s. $\rightsquigarrow$ **propositional equality**
- **Idea:** Let $x, y : A$. Then there is a type $(x =_A y)$ of *identifications* or *proofs* that x is equal to y (*formation*).
- In a topological picture, we could imagine $p : (x =_A y)$ to be a *path* from x to y (more on this later).
- For any $x : A$ there should be a term $\mathrm{refl}_x : (x =_A x)$ (*introduction*).

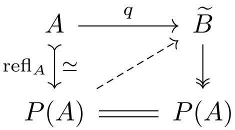
$$\frac{\Gamma \vdash A}{\Gamma \vdash x, y : A \vdash (x =_A y)}\ \text{Id-Form} \qquad \frac{\Gamma \vdash A}{\Gamma, x : A \vdash \mathrm{refl}_x : (x =_A x)}\ \text{Id-Intro}$$

Identity type: elimination and computation

- **Q:** How do we *eliminate* out of $(x =_A y)$?
- **A:** Via induction principle!
- **Idea:** Identity types are freely generated by the reflexivity terms.
- **Identity elimination:** Given a type B depending on $x, y : A$ and $p : x =_A y$, to give a term in $B(x, y, p)$ we can assume $y \equiv x$ and $p \equiv \mathrm{refl}_x$:

$$\frac{\Gamma \vdash A \qquad \Gamma, x : A, y : A, p : x =_A y \vdash B(x, y, p)}{\Gamma \vdash \mathrm{ind}_{=_A} : \prod_{a:A} B(a, a, \mathrm{refl}_a) \to \prod_{x,y:A} \prod_{p:(x=_A y)} B(x, y, p)} \text{ Id-Elim}$$

  with *computation* rule: $\mathrm{ind}_{=_A}(q, a, a, \mathrm{refl}_a) \equiv q$
- Model-categorical semantics:



Path induction

From identity elimination we can prove:

Theorem ((based) path induction)

Fix $b : B$. *For* $P : \left(\sum_{x:B}(b =_B x)\right) \to \mathcal{U}$ *we have an equivalence:*

$$\left(\prod_{x:B}\prod_{p:b=_B x} P(x,p)\right) \underset{\mathrm{ind}}{\overset{\mathrm{ev}_{\mathrm{refl_b}}}{\rightleftarrows}} P(b, \mathrm{refl}_b) \qquad (\simeq)$$

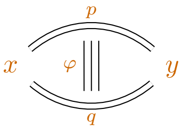
The Curry–Howard–Voevodsky interpretation[1]

Type theory	Logic	Set theory	Homotopy theory
A	proposition	set	space/homotopy type
$x : A$	witness/realizer	element	point
$\mathbf{0},\mathbf{1}$	$\bot$, $\top$	$\emptyset$, $\{\emptyset\}$	$\emptyset$, $*$
$A + B$	$A \vee B$	disjoint union	coproduct space
$A \times B$	$A \wedge B$	set of ordered pairs	product space
$A \to B$	$A \Rightarrow B$	set of functions	function space
$x : A \vdash B(x)$	predicate $B(x)$	family of sets	fibration
$x : A \vdash b : B(x)$	conditional proof	choice of elements	section
$\Sigma_{x:A} B(x)$	$\exists\, x.B(x)$	disjoint sum	total space
$\Pi_{x:A} B(x)$	$\forall\, x.B(x)$	product	space of sections
$p : (x =_A y)$	$x = y$	$x = y$	path $x \rightsquigarrow y$ in A

[1] Table based on: Emily Riehl *The synthetic theory of ∞-categories vs the synthetic theory of ∞-categories*, Presentation at Vladimir Voevodsky Memorial Conference, IAS, Princeton, NJ, USA, 2018.

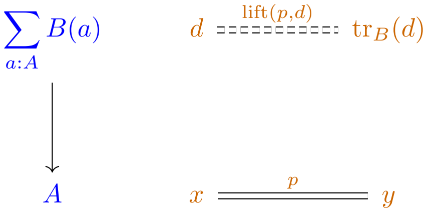
Types as ∞-groupoids

- Can define *symmetry/inversion* $(-)^{-1} : \prod_{x,y:A} (x =_A y) \to (y =_A x)$ and *transitivity/composition* $* : \prod_{x,y,z:A} (x =_A y) \to (y =_A z) \to (x =_A z)$.
- One can also show some expected laws, namely **associativity**, **neutrality**, and **inversion**, *e.g.*
  $(p * q) * r =_{(x=_A z)} p * (q * r), \quad \mathrm{refl}_y * p =_{(x=_A y)} p, \quad p^{-1} * p =_{(x=_A x)} \mathrm{id}_x, \ldots$
- These are known as the *groupoid laws*.
- They arise as *higher identities/homotopies* $\varphi : p =_{(x=_A y)} q$ for $p, q : (x =_A y)$:



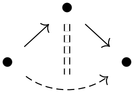
Dependent types as fibrations

- Identity proofs $p : (x =_A y)$ are a kind of "path".
- Indeed, dependent types behave well w.r.t. paths in the base.
- Namely, every dependent type supports a notion of **path transport**. $\implies$ Dependent types are **fibrations**.
- For $a : A \vdash B$, we can define a function $\mathrm{tr}_A : \prod_{x,y:A} \prod_{p:(x=_A y)} B(x) \to B(y)$.
- Again by path induction, with $\mathrm{tr}_B(x, x, \mathrm{refl}_x) :\equiv \mathrm{id}_{B(x)}$.
- Indeed, this is connected with a synthetic notion of **path lifting**:

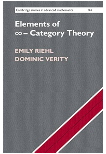


The concept of $(\infty, 1)$-category

- $(\infty, 1)$**-categories:** weak composition of 1-morphisms given uniquely *up to contractibility*



- How to express this in HoTT?
- *Problem:* We have path types $(a =_A b)$, but what about directed hom types $(a \to_A b)$?
- Several possible type-theoretic frameworks, *e.g.* by Warren, Licata–Harper, Annenkov–Capriotti–Kraus–Sattler, Nuyts, North, Weaver–Licata,. . .
- Other synthetic theories: Riehl–Verity, Cisinski–Cnossen–Nguyen–Walde, Martini–Wolf
- But their connection to traditional ∞-category theory is less clear.
- **In our work:** Riehl–Shulman's *simplicial type theory* (2017). Also heavily influenced by Riehl–Verity's ∞-cosmos theory (2013-2021-. . .).




Higher Structures 1(1):116–193, 2017.

HIGHER STRUCTURES

A type theory for synthetic ∞-categories

Emily Riehl[a] and Michael Shulman[b]

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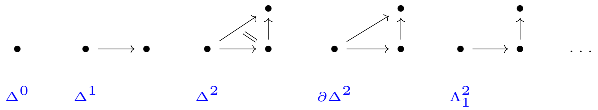
Simplicial HoTT

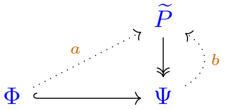
1. **Simplicial HoTT:** Extension of HoTT by Riehl–Shulman '17
2. add strict shapes



3. add extension types (due to Lumsdaine–Shulman, cf. Cubical Type Theory):

Input:

- shape inclusion $\Phi \hookrightarrow \Psi$
- family $P : \Psi \to \mathcal{U}$
- partial section $a : \Pi_{t:\Phi} P(t)$

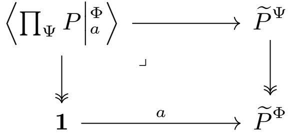


$\rightsquigarrow$

Extension type $\left\langle \prod_\Psi P \middle|^{\Phi}_{a} \right\rangle$

with terms $b : \Pi_\Psi P$ such that $b|_\Phi \equiv a$.

Semantically:



Hom types I

Definition (Hom types, [RS17])

Let B be a type. Fix terms $a, b : B$. The type of *arrows in* B *from* a *to* b is the extension type

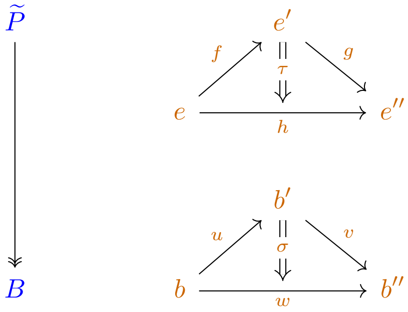
$$\hom_B(a,b) :\equiv (a \to_B b) :\equiv \left\langle \Delta^1 \to B \middle|^{\partial\Delta^1}_{[a,b]} \right\rangle.$$

Definition (Dependent hom types, [RS17])

Let $P : B \to \mathcal{U}$ be family. Fix an arrow $u : \hom_B(a,b)$ in B and points $d : P\,a$, $e : P\,b$ in the fibers. The type of *dependent arrows in* P *over* u *from* d *to* e is the extension type

$$\mathrm{dhom}_{P,u}(d,e) :\equiv (d \to^P_u e) :\equiv \left\langle \prod_{t:\Delta^1} P(u(t)) \middle|^{\partial\Delta^1}_{[d,e]} \right\rangle.$$

Hom types II



Segal, Rezk, and discrete types

We can now define synthetic $(\infty,1)$-categories using shapes and extension types:

Definition (Synthetic $(\infty,1)$-categories, [RS17])

- **Synthetic pre-$(\infty,1)$-category *aka* Segal type:** types A with *weak composition*, *i.e.*:

$$\iota : \Lambda_1^2 \hookrightarrow \Delta^2 \rightsquigarrow A^\iota : A^{\Delta^2} \xrightarrow{\simeq} A^{\Lambda_1^2} \qquad \text{(Joyal).}$$

- **Synthetic $(\infty,1)$-category *aka* Rezk type:** Segal types A satisfying *Rezk completeness/local univalence*, *i.e.*

$$\mathrm{idtoiso}_A : \Pi_{x,y:A}(x =_A y) \xrightarrow{\simeq} \mathrm{iso}_A(x,y).$$

- **Synthetic ∞-groupoid *aka* discrete type:** types A such that *every arrow is invertible*, *i.e.*

$$\mathrm{idtoarr}_A : \Pi_{x,y:A}(x =_A y) \xrightarrow{\simeq} \hom_A(x,y).$$

Adequate semantics of synthetic ∞-category theory

Theorem (Shulman ’19, Riehl–Shulman ’17)

1. *Every ∞-topos admits a model of HoTT.*
2. *Every ∞-topos of simplicial objects admits a model of sHoTT, with weakly stable extension types.*

Theorem (W ’21)

Extension types are strictly substitution-stable.

Extension types have become of independent interest in type and programming language theory.

Corollary

1. *Synthetic ∞-category theory interprets to ordinary ∞-category theory.*
2. *Synthetic ∞-category theory interprets to internal ∞-category theory (cf. Martini–Wolf, Cisinski–Ngyuen–Walde–Cnossen, Rasekh, Stenzel).*

Functors and natural transformations

- Segal types have **categorical structure**: composition $g \circ f$, identities id_x, and homotopies
$$h \circ (g \circ f) = (h \circ g) \circ f, \quad \mathrm{id}_y \circ f = f, \quad f \circ \mathrm{id}_x = f.$$
- Any map $f : A \to B$ between Segal types is automatically a **functor**.
- For $f, g : A \to B$ define the type of **natural transformations** as
$$(f \Rightarrow g) :\equiv \hom_{A \to B}(f, g) :\equiv \left\langle \Delta^1 \to (A \to B) \Big|^{\partial\Delta^1}_{[f,g]} \right\rangle.$$
- One can then *prove* that for $\varphi : (f \Rightarrow g)$ any arrow $u : x \to_A y$ gives rise to the expected naturality square:
$$\begin{array}{ccc} fx & \xrightarrow{\varphi_x} & gx \\ {\scriptstyle fu}\downarrow & & \downarrow{\scriptstyle gu} \\ fy & \xrightarrow[\varphi_y]{} & gy \end{array}$$

Fibered Yoneda lemmas

- Riehl–Shulman had introduced functorial type families, but only valued in groupoids.
- The fundamental principle of category theory is the **Yoneda lemma**: characterization of functorial families by maps from base elements into it

Theorem (Yoneda lemma for discrete covariant families, Riehl–Shulman '17)

For a covariant discrete family $P : b \downarrow B \to \mathcal{U}$ *evaluation at the identity is an equivalence:*

$$\mathrm{ev}_{\mathrm{id}_b} : \left(\prod_{b \downarrow B} P \right) \xrightarrow{\simeq} P(\mathrm{id}_b)$$

Theorem (Yoneda lemma for cocartesian families, Buchholtz–W' 21)

For a cocartesian family $P : b \downarrow B \to \mathcal{U}$*, evaluation at the identity is an equivalence:*

$$\mathrm{ev}_{\mathrm{id}_b} : \left(\prod_{b \downarrow B}^{\mathrm{cocart}} P \right) \xrightarrow{\simeq} P(\mathrm{id}_b)$$

Covariant type families

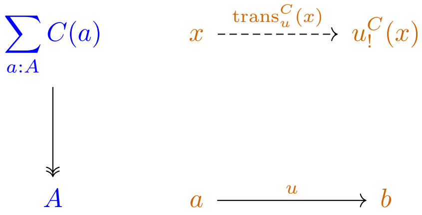
Definition (Covariant family, [RS17])

Let $C : A \to \mathcal{U}$ be a family. It is *covariant* if and only if for all $a, b : A$, arrows $u : (a \to_A b)$ and points $x : C(a)$ the type

$$\sum_{y : C(b)} (x \to^C_u y)$$

is contractible.

This give a synthetic analogue of discrete covariant or *left* fibrations:



Fibered Yoneda lemma as directed path induction

Theorem

Fix $b : B$. *For* $P : \left(\sum_{x:B}(b =_B x)\right) \to \mathcal{U}$ *we have an equivalence:*

$$\left(\prod_{x:B}\prod_{p:b=_B x} P(x,p)\right) \underset{\text{ind}_b^P}{\overset{\text{ev}_{\text{refl}_b}}{\rightleftarrows}} P(b, \text{refl}_b) \qquad \simeq$$

Theorem ((dependent) Yoneda Lemma for covariant families, [RS17])

Let B *be a Segal type, and fix* $b : B$. *For a covariant type family* $P : \left(\sum_{x:B}(b \to_B x)\right) \to \mathcal{U}$, *we have an equivalence:*

$$\left(\prod_{x:B}\prod_{p:b\to_B x} P(x,p)\right) \underset{\mathbf{y}_b^P}{\overset{\text{ev}_{\text{id}_b}}{\rightleftarrows}} P(b, \text{id}_b) \qquad \simeq$$

Fibered Yoneda lemma: proof idea

Theorem (Yoneda Lemma for covariant families, [RS17])

Let A be a Segal type, and $a : A$ any term. For a covariant type family $C : A \to \mathcal{U}$, we have an equivalence:

$$\mathrm{evid}_a^C : \left(\prod_{x:A} \hom_A(a,x) \to C(x) \right) \xrightarrow{\simeq} C(a)$$

- The inverse map is given by

$$\mathbf{y}_a^C : C(a) \to \left(\prod_{x:A} \hom_A(a,x) \to C(x) \right), \quad \mathbf{y}_a^C(u)(x)(f) :\equiv f_! u$$

- Proof "simply" follows from naturality properties and covariance of $\hom_A(a,-)$.
- There also exists a *dependent version*.
- Both have been formalized in Kudasov's new proof assistant Rzk.
- Cocartesian and other generalizations due to Buchholtz–W and W have been proven, but formalization is WIP.

The Rzk proof assistant

- SRC
  - hott
    - 06-contractible.rzk.md
    - 07-fibers.rzk.md
    - 08-families-of-maps.rzk.md
    - 09-propositions.rzk.md
    - 10-trivial-fibrations.rzk.md
  - simplicial-hott
    - 03-simplicial-type-theory.rzk.md
    - 04-extension-types.rzk.md
    - 05-segal-types.rzk.md
    - 06-2cat-of-segal-types.rzk.md
    - 07-discrete.rzk.md
    - 08-covariant.rzk.md
    - 09-yoneda.rzk.md
    - 10-rezk-types.rzk.md
    - 12-cocartesian.rzk.md M
- OUTLINE
- TIMELINE

simplicial-hott > 05-segal-types.rzk.md

~~~~
```rzk
#assume funext : FunExt
#assume extext : ExtExt
```

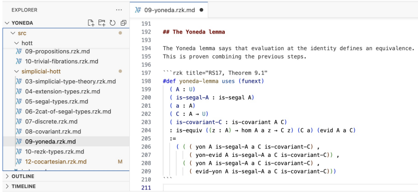

## Hom types

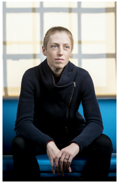
Extension types are used to define the type of arrows
between fixed terms:

```rzk title="RS17, Definition 5.1"
#def hom
  ( A : U)
  ( x y : A)
  : U
  :=
    ( t : Δ¹) →
    A [ t ≡ 0₂ ↦ x ,  -- the left endpoint is exactly x
        t ≡ 1₂ ↦ y]   -- the right endpoint is exactly y
```
~~~~

Formalizing ∞-categories in Rzk

- Kudasov has developed the Rzk proof assistant, implementing sHoTT: `https://rzk-lang.github.io/`
- Using Rzk we initiated the first ever formalizations of ∞-category theory.
- In spring 2023, with Kudasov and Riehl we formalized the (discrete fibered) Yoneda lemma of ∞-category theory: `https://emilyriehl.github.io/yoneda/`
- alongside many other results
- Many proofs in this ∞-dimensional setting *easier* than in dimension 1!
- Formalization helped find a mistake in original paper
- More students & researchers joined us developing a library for ∞-category theory: `https://rzk-lang.github.io/sHoTT/` **Join us!**





Synthetic ∞-category theory in sHoTT

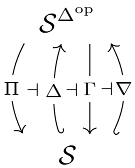
- Functors, natural transformations, discrete fibrations & fibered Yoneda lemma, adjunctions (Riehl–Shulman '17)
- Cartesian fibrations (Buchholtz–W '21) & generalizations (W '21)
- Limits and colimits (Bardomiano '22)
- Conduché fibrations (Bardomiano '24)
- Proof assistant `Rzk` (Kudasov '23) and formalization of fibered Yoneda lemma (Kudasov–Riehl–W '23)
- sHoTT library and more formalizations (Abounegm, Bakke, Bardomiano, Campbell, Carlier, Chatzidiamantis-Christoforidis, Ergus, Hutzler, Kudasov, Maillard, Martínez, Pradal, Rasekh, Riehl, F. Verity, Walde, W '23–)

But many desiderata missing!

opposite categories, categories $\mathcal{S}$ and Cat, presheaves & Yoneda embedding, higher algebra, ...

Multimodal type theory

- **Multi-modal dependent type theory (MTT)** to the rescue! (Gratzer–Kavvos–Nuyts–Birkedal '20)
- start from a *cubical* outer layer, augmented by an instance of MTT
- the added modal operators: **simplicial localization** $\boxslash$, **opposite** op, **twisted arrows** tw **(groupoid) core/discretization** $\flat \dashv$ **codiscretization** $\sharp$, **path type** $(-)^{\mathbb{I}} \dashv$ **amazing right adjoint** $(-)_{\mathbb{I}}$
- plus axioms about the interaction between the simplicial and modal structure
- This unlocks a whole new range of constructions
- We call the ensuing type theory *triangulated type theory*



$$\textstyle\int :\equiv L_{\mathbb{I}} \equiv \Delta \circ \Gamma \text{ (Rijke–Shulman–Spitters '20)}$$

$$\flat :\equiv \Delta \circ \Gamma$$

$$\sharp :\equiv \nabla \circ \Gamma$$

See also work on cohesive ∞-toposes by Schreiber ('13), Shulman ('18), Myers–Riley ('23), as well as internal universes via a tiny interval by Licata–Orton–Pitts-Spitters ('18) and Riley ('24).

Mode theory

One single mode m with the modalities for the tiny interval, cohesion, and direction. Some intuitions:

- **Opposite** op: $\langle \mathrm{op} \mid A \rangle$ has its n-simplices reversed
- **Discretization/core** $\flat$: $\langle \flat \mid A \rangle \to A$ is the maximal subgroupoid of A
- **Codiscretization** $\sharp$: $A \to \langle \sharp \mid A \rangle$ is localization at $\partial\Delta^n \to \Delta^n$ (for crisp closed types)
- **Twisted arrows** tw: $\langle \mathrm{tw} \mid A \rangle$ has as n-simplices:

$$\begin{array}{ccccccccc} a_n & \longleftarrow & \dots & \longleftarrow & a_2 & \longleftarrow & a_1 & \longleftarrow & a_0 \\ \downarrow & & & & & & & & \\ a_{n+1} & \longrightarrow & \dots & \longrightarrow & a_{2n-2} & \longrightarrow & a_{2n-1} & \longrightarrow & a_{2n} \end{array}$$

Mode theory:

$$\flat \circ \flat = \flat \circ \mathrm{op} = \flat \circ \sharp = \mathrm{tw} \circ \flat = \mathrm{op} \circ \flat = \flat \qquad \sharp \circ \sharp = \sharp \circ \flat = \sharp \circ \mathrm{op} = \mathrm{op} \circ \sharp = \sharp$$

$$\mathrm{op} \circ \mathrm{op} = \mathrm{id} \qquad \varepsilon : \flat \to \mathrm{id} \qquad \eta : \mathrm{id} \to \sharp$$

$$\eta \cdot \sharp = \sharp \cdot \eta = \mathrm{id} : \sharp \to \sharp \qquad \flat \cdot \eta = \mathrm{id} : \flat \to \flat$$

plus some coherence conditions for tw

Axioms for triangulated type theory I

Axiom (Interval $\mathbb{I}$)

There is a bounded distributive lattice $(\mathbb{I} : \mathrm{Set}, 0, 1, \vee, \wedge)$

Axiom (Path type former as modality)

The path type $(-)^{\mathbb{I}}$ *is presented by a modality* $\mathfrak{p}$.

Axiom (Crisp induction)

Modalities commute with path types: for every μ, *the map* $\mathrm{mod}_\mu(a) = \mathrm{mod}_\mu(b) \to \langle \mu \mid a = b \rangle$ *is an equivalence.*

Axiom (Reversal on $\mathbb{I}$)

There is an equivalence $\neg : \langle \mathrm{op} \mid \mathbb{I} \rangle \to \mathbb{I}$ *which swaps* 0 *for* 1 *and* $\wedge$ *for* $\vee$.

Axioms for triangulated type theory II

Axiom ($\mathbb{I}$ detects discreteness)

If $A :_\flat \mathcal{U}$ *then* $\langle \flat \mid A \rangle \to A$ *is an equivalence if and only if* $A \to (\mathbb{I} \to A)$ *is an equivalence.*

Axiom (Global points of $\mathbb{I}$)

The map $\mathbb{B} \to \mathbb{I}$ *is injective and induces an equivalence* $\mathbb{B} \simeq \langle \flat \mid I \rangle$.

Axiom (Cubes separate)

$f :_\flat A \to B$ *is an equivalence if and only if the following holds:*

$$\Pi_{n :_\flat \mathbb{N}} \operatorname{isEquiv} \left(f_* : \langle \flat \mid \mathbb{I}^n \to A \rangle \to \langle \flat \mid \mathbb{I}^n \to B \rangle \right)$$

Axiom (Simplicial stability)

If $A :_\flat \mathcal{U}$ *then for all* $n :_\flat \mathbb{N}$ *the following map is an equivalence:*

$$\eta_* : \langle \flat \mid \Delta^n \to A \rangle \to \langle \flat \mid \Delta^n \to \boxslash A \rangle$$

Axioms for triangulated type theory III

Axiom (Twisted arrows)

For each $n :_\flat \mathbb{N}$ *there is a map* $\xi :_\flat \Delta^n \to \langle \mathrm{tw} \mid \Delta^{2n+1} \rangle$*, satisfying some naturality conditions, inducing an equivalence* $\xi^* :_\flat \langle \flat \mid \Delta^{2n+1} \to C \rangle \xrightarrow{\simeq} \langle \flat \mid \Delta^n \to \langle \mathrm{tw} \mid C \rangle \rangle$.

Axiom (Blechschmidt duality)

Let $\mathbb{I} \to A$ *be a finitely presented* $\mathbb{I}$*-algebra, i.e.,* $A \simeq \mathbb{I}[x_1, \ldots, x_n]/(r_1 = s_1, \ldots, r_n = s_n)$*, then the evaluation map is an equivalence:*

$$\mathrm{ev} \equiv \lambda a, f.f(a) : A \simeq (\hom_{\mathbb{I}}(A, \mathbb{I}) \to \mathbb{I})$$

Versions of the latter axiom appear in synthetic differential geometry (Kock–Lawvere axioms),[2] synthetic algebraic geometry,[3] and synthetic domain theory[4]

[2] A. Kock: *A simple axiomatics for differentiation*, Math. Scand. 40.2 (1977): 183–193 (JSTOR)

[3] F. Cherubini, T. Coquand, M. Hutzler: *A foundation for synthetic algebraic geometry*, Math. Struct. Comp. Sci. (2023): 1–46, doi:10.1017/S0960129524000239

[4] J. Sterling, L. Ye: *Domains and classifying topoi*, (2025): 1–47arXiv:2505.13096

Simplicial vs cubical models

Theorem (Kapulkin–Voevodsky '18, Sattler '18, Streicher–W '19)

Simplicial sets (spaces) are an essential sub(-∞-)topos of cubical sets (spaces).

Internally, a cubical type A is **simplicial** if

$$\mathrm{isSimp}(A) :\equiv \prod_{i,j:\mathbb{I}} \mathrm{isEquiv}(A^! : A \to A^{i \leq j \vee j \leq i}).$$

This defines a lex modality à la Rijke–Shulman–Spitters.

Applications of duality I

Lemma (Phoa's principle; ind. Pugh and Sterling)

$(\mathbb{I} \to \mathbb{I}) \simeq \Delta^2 \to \mathbb{I} \times \mathbb{I}$

Proof.

We have $\hom_{\mathbb{I}}(\mathbb{I}[x], \mathbb{I}) \simeq \mathbb{I}$. Thus, by duality, $\mathrm{ev} : \mathbb{I}[x] \simeq (\mathbb{I} \to \mathbb{I})$. Then it suffices to show that $\langle \mathrm{ev}_0, \mathrm{ev}_1 \rangle : \mathbb{I}[x] \to \mathbb{I} \times \mathbb{I}$ factors through Δ^2. Finrally, we get an equivalence $\mathbb{I}[x] \simeq \Delta^2$ by using the normal forms $p(x) = p(0) \vee x \wedge p(1)$. □

Lemma (Generalized Phoa's principle)

- $(\mathbb{I}^n \to \mathbb{I}) \simeq \mathbf{Pos}(\mathbb{B}^n, \mathbb{I})$
- $(\Delta^n \to \mathbb{I}) \simeq \mathbf{Pos}([0 \leq \ldots \leq n], \mathbb{I})$

See also: L. Pugh, J. Sterling: *When is the partial map classifier a Sierpiński cone?*, LICS 2025, arxiv:2504.06789

Applications of duality II

Theorem

$\mathbb{I}$ *is simplicial.*

Proof.

Have to show $\mathbb{I} \simeq ((i \leq j \vee j \leq i) \to \mathbb{I})$. By the generation axiom, it suffices to show for $f, g :_\flat \mathbb{I}^n \to \mathbb{I}$ that:

$$\langle \flat \mid \mathbb{I}^n \to \mathbb{I} \rangle \simeq \langle \flat \mid \{\vec{x} : \mathbb{I}^n \mid f(\vec{x}) \leq g(\vec{x}) \vee g(\vec{x}) \leq f(\vec{x})\} \to \mathbb{I} \rangle$$

By the generalized Phoa principle, to extend a function in the codomain amounts to defining it for all $\vec{x} : \mathbb{B}^n$. Indeed, $f(\vec{x}) \leq g(\vec{x}) \vee g(\vec{x}) \leq f(\vec{x})$.

$\square$

For an alternative proof, see M. D. Williams: *Projective Presentations of Lex Modalities*, arXiv:2501.19187.

Applications of duality III

Theorem

Δ^n *is a category.*

Proof.

Rezk-completeness is clear (no nontrivial isos). Simplicialness is clear (retract of $\square^n$). Segalness follows using (generalized) Phoa principle. □

Applications of duality IV

Theorem

If $A :_\flat \mathcal{U}$ is discrete then A is simplicial.

Proof.

By discreteness and cubical generation it suffices to show that f.a. $p, q : \mathbb{I}[\vec{x}]$ and $\varphi(\vec{x}) :\equiv p(\vec{x}) \leq q(\vec{x}) \vee q(\vec{x}) \leq p(\vec{x})$ that:

$$A \simeq (\varphi(\vec{x}) \to A)$$

For this, we construct a "homotopy" h from $\lambda \vec{x}.0$ to $\varphi(\vec{x})$. The claim then follows using Phoa's principle:

$$h(\vec{x}, t) :\equiv \vec{x} \wedge t$$

□

After Gratzer, using $\flat \dashv \sharp$ one can prove that $\mathbb{N}$ is discrete.

Corollary

$\mathbb{B}$ and $\mathbb{N}$ are both discrete and simplicial, i.e., groupoids.

Warning: The theorem is *false* for Rezk in place of discrete types, e.g. $\Delta^2 \amalg_{\Delta^1} \Delta^2$.

Towards the universe of spaces

- Covariant families have **transport**: $(-)_! : \prod_{a,b:X} (a \to_X b) \to A(a) \to A(b)$
- If X is Segal, then each fiber $A(a)$ is discrete.
- Can we take $\sum_{A:\mathcal{U}} \mathrm{isCov}(A)$?
- **No**: $\mathrm{isCov}(A)$ just means that A is discrete; doesn't see variance.
- Need a predicate that yields covariance over all possible contexts.
- **Solution: Amazing fibrations** due to M. Riley (2024): *A Type Theory with a Tiny Object* , arXiv:2403.01939; based on Licata–Orton–Pitts–Spitters '18 (which was used for similar purposes by Weaver–Licata '20)

Amazingly covariant families

- Consider $\text{isCov}(A : \square \to \mathcal{U}) \simeq \prod_{x:A(0)} \text{isContr}\left(\sum_{y:A(1)} (x \to_\alpha y)\right)$, where $\alpha : \text{hom}_\square(0,1)$.
- This gives a predicate $\text{isCov}_\square : \mathcal{U}^\square \to \text{Prop}$.

Definition (Amazingly covariant types)

Let $A : \mathcal{U}$ be a type. It is *amazingly covariant* if and only if the following proposition is inhabited:

$$\text{isACov}(A) :\equiv \big(\text{isCov}_\square(\lambda i.A^\eta(i))\big)_\square,$$

where A^η is the image of A under the unit $\eta_\mathcal{U} : \mathcal{U} \to (\mathcal{U}^\square)_\square$.

The universe of spaces

The simplicial objects give rise to the (simplicial) subuniverse of simplicial types:

$$\mathcal{U}_{\boxslash} :\equiv \sum_{A:\mathcal{U}} \mathrm{isSimp}(A)$$

Definition

We call $\mathcal{S} :\equiv \sum_{A:\mathcal{U}_{\boxslash}} \mathrm{isACov}(A)$ the **universe of spaces**.

Theorem

1. *The universe $\mathcal{S}$ is a synthetic ∞-category whose terms are ∞-groupoids.*
2. *$\mathcal{S}$ classifies amazingly covariant families in $\mathcal{U}_{\boxslash}$.*
3. *$\mathcal{S}$ is closed under Σ, identity types, and finite (co)limits.*
4. *$\mathcal{S}$ is **directed univalent**:*

$$\mathrm{arrtofun} : (\Delta^1 \to \mathcal{S}) \simeq \left(\sum_{A,B:\mathcal{S}} (A \to B) \right)$$

Equivalence lemma

Theorem

Assume maps $f, g : \Delta^1 \to \mathcal{S}$ *and a natural transformation* $\alpha : \prod_{x:\Delta^1} f(x) \to g(x)$*. Then* α *is a family of equivalences if and only if* $\alpha(0)$ *and* $\alpha(1)$ *are equivalences.*

$$\begin{array}{ccc} f\,0 & \xrightarrow[\cong]{\alpha\,0} & g\,0 \\ \downarrow & & \downarrow \\ f\,1 & \xrightarrow[\alpha\,1]{\cong} & g\,1 \end{array}$$

For the proof, we need the axiom that cubes detect equivalences:

$$\Big(\prod_{n:\mathrm{Nat}} \langle \flat \mid \mathbb{I}^n \to A \rangle \simeq \langle \flat \mid \mathbb{I}^n \to B \rangle\Big) \to (A \simeq B)$$

We can also prove a generalization of the equivalence lemma to Δ^n.

Directed univalence

1. Since $\mathcal{S}$ classifies (amazingly) covariant families, there is a map

$$\mathrm{arrtofun} :\equiv \lambda F.(F\,0, F\,1, \alpha_!^F : F\,0 \to F\,1) : (\Delta^1 \to \mathcal{S}) \to \Big(\sum_{A,B:\mathcal{S}} (A \to B)\Big).$$

2. In the other direction, we consider the **mapping cone/directed glue type** (cf. cubical type theory and Weaver–Licata '20):

$$\mathrm{Gl} :\equiv A, B, f.\lambda i. \sum_{b:B} (i = 0) \to f^{-1}(b) : \Big(\sum_{A,B:\mathcal{S}} (A \to B)\Big) \to (\Delta^1 \to \mathcal{S})$$

3. We show that they form an inverse pair making crucial use of the equivalence lemma.
4. Segalness of $\mathcal{S}$ is using similar arguments, but in higher dimensions.

Application: directed structure identity principle (DSIP)

Theorem (DSIP for pointed spaces)

Let $\mathcal{S}_* :\equiv \sum_{A:\mathcal{S}} A$*. Then for* $(A,a),(B,b) : \mathcal{S}_*$ *we have:*

$$\mathrm{hom}_{\mathcal{S}_*}((A,a),(B,b)) \simeq \sum_{f:A\to B} f(a) = b$$

Theorem (DSIP for monoids)

Consider the type (category!) of (set-)monoids

$$\mathrm{Monoid} :\equiv \sum_{A:\mathcal{S}_{\leq 0}} \sum_{\varepsilon:A} \sum_{\cdot:A\times A} \mathrm{isAssoc}(\cdot) \times \mathrm{isUnit}(\cdot, \varepsilon).$$

Then homomorphisms from $(A, \varepsilon_A, \cdot_A, \alpha_A, \mu_A)$ *to* $(B, \varepsilon_B, \cdot_B, \alpha_B, \mu_B)$ *correspond to set maps* $A \to B$ *compatible with multiplication and units.*

Towards synthetic higher algebra

We can internally define presheaf categories $\mathrm{PSh}(C) :\equiv \langle \mathfrak{o} | C \rangle \to \mathcal{S}$.

Definition (∞-monoids)

The category Mon_∞ of ∞*-monoids* is the full subcategory[a] of $\mathrm{PSh}(\Delta)$ defined by the predicate

$$\varphi(X :_\flat \mathrm{PSh}(\Delta)) :\equiv \prod_{n:\mathrm{Nat}} \mathrm{isEquiv}(\langle X(\iota_k)_{k<n} \rangle : X(\Delta^n) \to X(\Delta^1)^n)$$

[a] need the codiscrete modality $\sharp$

This encodes the structure of a homotopy-coherent monoid. Multiplication is given through

$$\mu_X : X(\Delta^1) \simeq X(\Delta^1)^2 \to X(\Delta^1).$$

Definition (∞-groups)

The category Grp_∞ of ∞*-groups* is the full subcategory of Mon_∞ defined by the predicate

$$\varphi(X :_\flat \mathrm{Mon}_\infty) :\equiv \mathrm{isEquiv}(\lambda x, y. \langle x, \mu_X(x,y) \rangle : X(\Delta^1)^2 \to X(\Delta^1)^2)$$

One can show that both these categories satisfy the expcted DSIP.

The category of spectra

Definition (The category of spectra)

The type of *spectra* is defined as the limit (in the ambient universe)

$$\mathrm{Sp} :\equiv \varprojlim(\mathcal{S}_* \xleftarrow{\Omega} \mathcal{S}_* \xleftarrow{\Omega} \ldots).$$

Proposition

Sp *is a stable* ∞*-category and cocomplete.*

Functorial Yoneda lemma

Let C be a category. Using the twisted arrow modality $\mathfrak{t}$, we obtain the hom-bifunctor $\Phi : C \times \langle \mathfrak{o} | C \rangle \to \mathcal{S}$. We write $\mathbf{y}(c) :\equiv \Phi(-, c)$.
We now recover the synthetic ∞-categorical version of the "standard" Yoneda lemma:

Theorem (Yoneda lemma)

We have $\hom(\mathbf{y}(c), X) \simeq X(c)$, *naturally in each* $c : C$ *and* $X : \mathrm{PSh}(C)$.

Theorem (Density)

If $X :_\flat \mathrm{PSh}(C)$, *then* $X \simeq \varinjlim_{\langle \mathfrak{o} \mid \widetilde{X} \rangle} \mathbf{y} \circ \pi^{\mathrm{op}}$.

Universal property of presheaf categories

Theorem (Descent for presheaf categories)

Let $E :\equiv \mathrm{PSh}(A)$ *and* $F :_\flat C \to E$*, then* $E/\varinjlim_{c:C} F(c) \simeq \varprojlim_{c:C} E/F(c)$.

Theorem (Universal property of $\mathrm{PSh}(C)$)

$\mathrm{PSh}(C)$ *is the free cocompletion of* C*:* $\mathbf{y}^* : (\mathrm{PSh}(C) \to_{\mathrm{cc}} E) \to (C \to E)$

Kan extensions

The notion of Kan extensions subsumes all the other fundamental concepts of category. theory

– S. Mac Lane ’71

Definition (Kan extensions)

Given $f :_\flat C \to D$ and a category E, the left Kan extension lan_f is the left adjoint to $f^* : E^D \to E^C$.

Theorem (Colimit formula)

If E is cocomplete, then lan_f exists. For $X :_\flat C \to E$ it computes to
$\mathrm{lan}_f X\, d \simeq \varinjlim(C \times_D D/d \to C \to E)$

Quillen's Theorem A

Definition (Cofinal functors)

A functor $f :_\flat C \to D$ is *right cofinal* if for every $X :_\flat D \to \mathcal{S}$ the map $\varprojlim_D X \to \varprojlim_C X \circ f$ is an equivalence.

Theorem (Quillen's Theorem A)

A functor $f :_\flat C \to D$ is right cofinal if and only if $L_{\mathbb{I}}(C \times_D d/D) \simeq \mathbf{1}$ for each $d :_\flat D$.

Application to cocartesian fibrations

Theorem (Properness of cocartesian fibrations)

As below, if π are cocartesian and m is right cofinal then v is right cofinal:

$$\begin{array}{ccc} A \times_B E & \xrightarrow{\ v\ } & E \\ {\scriptstyle \xi}\downarrow & \lrcorner & \downarrow{\scriptstyle \pi} \\ A & \xrightarrow{\ u\ } & B \end{array}$$

Using Quillen's Theorem A and some localization theory we can give a new synthetic proof:

Proof.

We compute the fiber:

$$(A \times_B E) \times_E e/E \simeq A \times_B e/E \simeq A \times_B \left(\Sigma_{b':B}\Sigma_{f:(\pi(e) \to_B b')}(E_{b'}^{\Delta^1})\right)$$
$$\simeq \Sigma_{\langle a,f\rangle : A \times_B \pi(e)/B} \; f_! \, e/E_{u(a)}$$

Now, we have both $L_{\square}(A \times_B \pi(e)/B) \simeq \mathbf{1}$ and $L_{\square}(f_! \, e/E_{u(a)}) \simeq \mathbf{1}$. This suffices by a theorem in: E. Rijke, M. Shulman, B. Spitters (2020): *Modalities in homotopy type theory*. □

Outlook

1. Synthetic higher algebra
2. Universe of higher categories
3. Extend formalizations
4. ...

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