Towers of sublocales induced by cozero elements

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Some basic concepts in frame theory

Categories: **Frm** and **Loc**

A *frame* is a complete lattice L which satisfies

$$x \wedge \bigvee x_i = \bigvee (x \wedge x_i) \text{ for all } x, x_i \in L$$

Examples: Complete Boolean/Heyting algebras

The open sets $\mathcal{O}X$ of a topological space X

A *frame map* preserves finite meets and arbitrary joins.

Examples: lattice morphism between complete Boolean algebras

Preimage of continuous maps

$$\mathbf{Loc} = \mathbf{Frm}^{op}$$

Sublocales/quotient frames/nuclei/congruences

Every frame is a complete Heyting algebra: $x \to y = \bigvee\{z : z \wedge x \leq y\}$

Note that $z \wedge x \leq y$ iff $z \leq x \to y$ Galois connection

A subset S of L is a *sublocale* if

1. S is closed under $\bigwedge$
2. $x \to s \in S$ for all $s \in S$ and $x \in L$.

- S has the same meets as L
- $1 \in S$
- joins in S may be different from joins in L

The collection of all sublocales form a coframe: (arbitrary) meet is intersection.

Sublocales/quotient frames/nuclei/congruences

A frame map $h : L \longrightarrow M$ has a right adjoint:

- $h_*(x) = \bigvee\{a \in L \mid h(a) = x\}$ largest element mapped to x
- $h(h_*(x)) = x$

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If h is onto, $h_*[M] \cong M$,

- $h_*[M]$ is a sublocale of L $h_* \circ h$ is a nucleus

If S is a sublocale of L, define a frame map $h : L \longrightarrow S$

$$h(a) = \bigwedge\{s \in S \mid s \geq a\}$$

- h is an onto i.e. a frame quotient

Skeleton a.k.a. Booleanization: $\mathfrak{B}L$

a^* denotes the *pseudocomplement* of $a \in L$: "complement of closure"

- largest element disjoint from a
- $a^* = \bigvee\{x | x \wedge a = 0\} = a \to 0$
- $a^* = a^{***}$

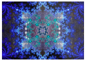
a is *dense* if $a^* = 0$ "closure" is the top

a^{**} denotes the *double pseudocomplement* "interior of closure"

$\mathfrak{B}L = \{a^{**} : a \in L\}$ is the Booleanization/skeleton "regular opens"

- complete Boolean algebra
- sublocale of L
- has no proper dense elements

Dense sublocales/quotient frames



S is a *dense* sublocale of L if $0 \in S$ — "closure" of S is L

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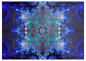

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Theorem (Isbell)

Every locale has a smallest dense sublocale namely $\mathfrak{B}L = \{x^{**} : x \in L\}$.

S is dense iff $\mathfrak{B}L \subseteq S$

No spatial analog

Cozero elements

Rather below relation: $b \prec a$ means that $b^* \vee a = 1$

- $a \in L$ is *cozero* if $a = \bigvee_n \{a_n | a_n \prec a\}$
- Coz L denotes all cozero elements of L
- largest regular sub-σ-frame of L

A frame L is *completely regular* if it is join-generated by Coz L

These may be described as images of "continuous" real-valued functions: $a \in \text{Coz}\, L$ iff $a = h(\mathbb{R} \smallsetminus \{0\})$ for some frame map $h : \mathcal{O}\mathbb{R} \longrightarrow L$.

The cozero tree rings of a (completely regular) frame

Kappa cozeros

For a regular cardinal κ:

- $a \in L$ is a *κ-cozero* if a is a κ-join of cozeros
- $\kappa\mathrm{Coz}\ L$ denotes the κ-cozero elements of L
- largest regular regular sub-κ-frame of L

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Cozero tower

Ascending sequence of regular sub-κ-frames of L:

$$\text{Coz}\, L = \omega_1\text{Coz}\, L \subseteq \cdots \subseteq \kappa\text{Coz}\, L \subseteq \cdots \subseteq \rho\text{Coz}\, L = L$$

The least such cardinal ρ is called the perfectly normal degree of L

- L is perfectly normal iff PN degree is ω_1 for example, $\mathcal{O}\mathbb{R}$

BLAST

κ-Lindelöf coreflections

A frame is κ-Lindelöf if every cover has a κ-sized subcover.

Theorem (Madden & Vermeer)

κ-Lindelöf completely regular frames are a coreflective (full) subcategory of completely regular frames

- denoted by $\mathscr{L}_\kappa L$: all κ-ideals of $\kappa\mathrm{Coz}\, L$ (downsets closed under κ-joins)
- "free" frame over $\kappa\mathrm{Coz}_L$ (as a κ-frame)
- coreflection map is given by join and is a frame quotient, so L may be identified as a sublocale of $\mathscr{L}_\kappa L$
- $\kappa\mathrm{Coz}\, L = \kappa\mathrm{Coz}\, \mathscr{L}_\kappa L$

The “Lindelöf” tower

- $\text{Coz}\, L = \text{Coz}\, \mathscr{L}_\kappa L$ for all κ
- $\kappa\text{Coz}\, L = \kappa\text{Coz}\, \mathscr{L}_\gamma L$ for all $\gamma \geq \kappa$

Ascending sequence of sublocales:

Lindelöf tower

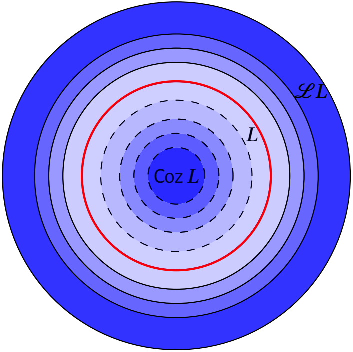
$$L = \mathscr{L}_\mu L \subseteq \cdots \subseteq \mathscr{L}_\gamma L \subseteq \cdots \subseteq \mathscr{L}_\kappa L \subseteq \cdots \subseteq \mathscr{L}_{\omega_1} L$$

where $\mu \geq \gamma \geq \kappa \geq \omega_1$

The least such cardinal μ is called the Lindelöf degree of L

- L is Lindelöf iff the Lindelöf degree $\mu \leq \omega_1$
- For example, the Lindelöf degree of $\mathcal{O}\mathbb{R}$ is ω_1

The cozero "tree rings" of a completely regular frame L



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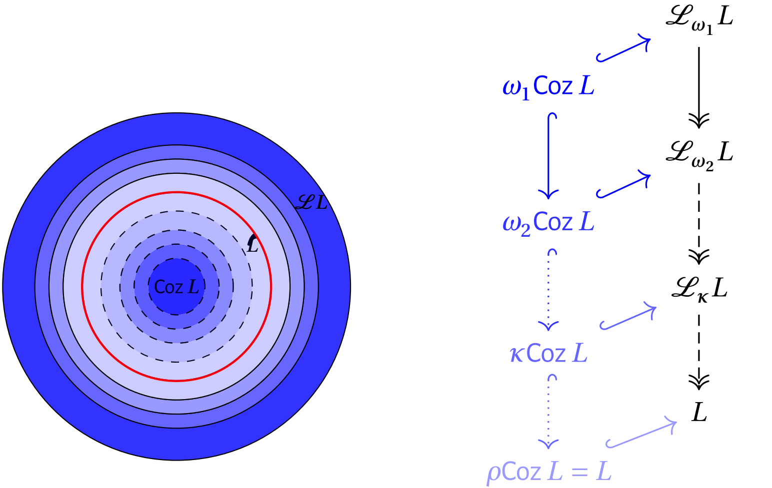

Bruns-Lakser completion of a regular κ-frame

Frames are the injective hulls of $\wedge$-semilattices.

- For any $\wedge$-semilattice A, the injective hull will be denoted by BL A: may be described as a sublocale of the frame of downsets
- If A has a least element, 0_A, then BL A is dense
- If A is a sub-$\wedge$-semilattice of a frame L, and A join-generates L, then BL A is a sublocale of L
- Moreover, if A has a least element, $\mathfrak{B}L$ is a sublocale of BL A
- if A is regular, then it is "proheyting", so the BL-completion is the normal (Dedekind MacNeille) completion

BL-tower of sublocales induced by cozeros

For each κ, consider BL $(\kappa\text{Coz}\ )$

- κCoz L is regular, hence "proheyting", so the BL-completion is the normal (Dedekind MacNeille) completion

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- $0 \in \kappa\text{Coz}\,L$ so $\mathfrak{B}L \subseteq$ BL (κCoz)
- If $\kappa\text{Coz}\,L \subseteq \lambda\text{Coz}\,L$, then BL (κCoz) is a sublocale of BL (λCoz)

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- $\kappa\text{Coz}\, L$ is a sub-$\wedge$-semilattice of L, join-generates L, so BL $\kappa\text{Coz}\, L$ is a sublocale of L
- $0 \in \kappa\text{Coz}\, L$ so $\mathfrak{B}L \subseteq \text{BL}\,(\kappa\text{Coz})$
- If $\kappa\text{Coz}\, L \subseteq \lambda\text{Coz}\, L$, then BL (κCoz) is a sublocale of BL (λCoz)
- If L is κ-perfectly normal (that is $L = \kappa\text{Coz}$), then $L = \text{BL}\,(\kappa\text{Coz})$

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- κCoz L is regular, hence "proheyting", so the BL-completion is the normal (Dedekind MacNeille) completion
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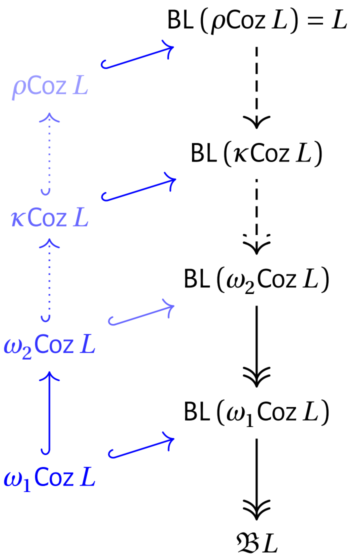
Ascending sequence of sublocales:

$$\text{BL}\,(\omega_1\text{Coz}\,L) \subseteq \cdots \subseteq \text{BL}\,(\kappa\text{Coz}\,L) \subseteq \cdots \subseteq \text{BL}\,(\rho\text{Coz}\,L) = L$$

Tower of BL-quotients



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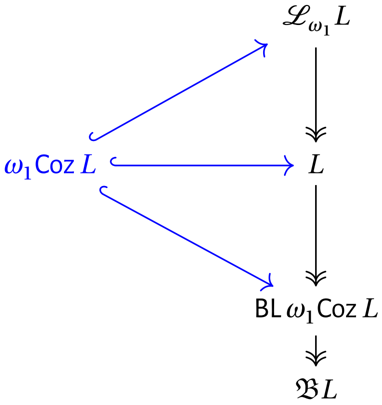

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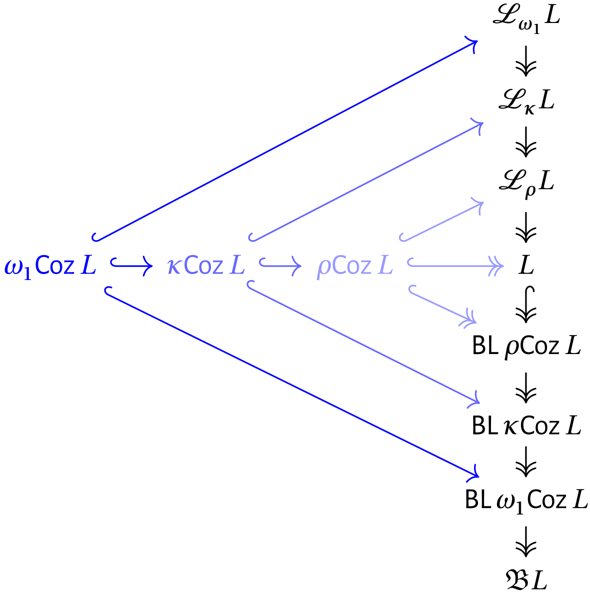
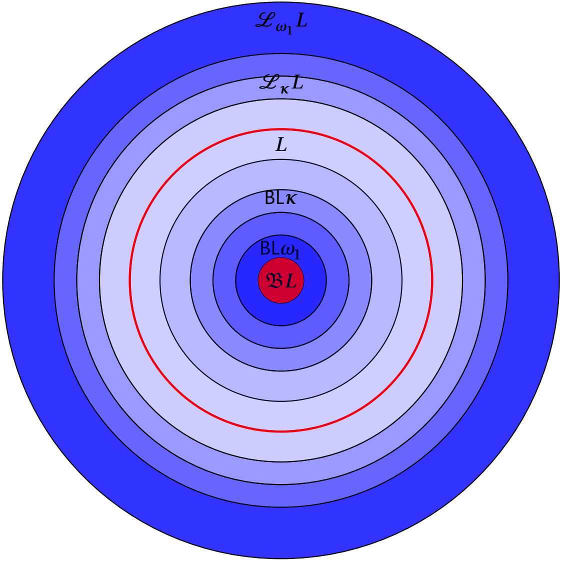
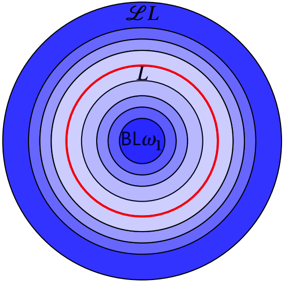


Cozero induced "tree rings" of a completely regular frame L



Some interesting "collapses"...



Total collapse $\iff \mathscr{L}_{\omega_1} L = L = \mathrm{BL}\,(\omega_1 \mathrm{Coz}\, L)$

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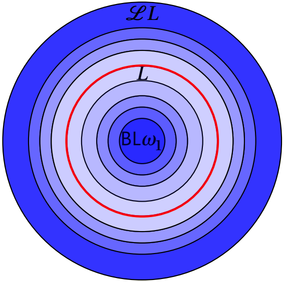

Total collapse $\iff \mathscr{L}_{\omega_1}L = L = \mathrm{BL}\,(\omega_1 \mathrm{Coz}\, L)$

$\iff \mathrm{BL}\,(\omega_1 \mathrm{Coz}\, L)$ **is Lindelöf**

Some interesting “collapses” wrt the skeleton

A frame L is ω_1*-hollow* if L has no proper dense cozero elements, that is, $\operatorname{Coz} L \subseteq \mathfrak{B}L$; aka almost P

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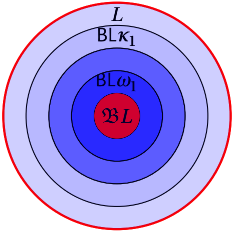

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A frame L is a *cozero frame* if Coz L is a frame.

(For example, all perfectly normal frames; converse is not true)

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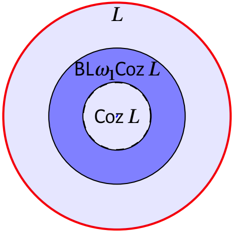

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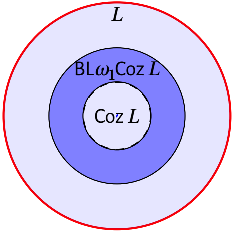

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This makes κCoz L is a sublocale of L

The "Hollowing out" tower of sublocales

In general, an element a in L can be "removed" by collapsing the frame:

- frame quotient $L \twoheadrightarrow {\downarrow}a$: $b \mapsto a \wedge b$
- corresponding sublocale is $\mathfrak{o}(a) = \{a \to b : b \in L\}$: each b in L is mapped to $a \to b$

Theorem (Ball, Hager, WW)

λ-hollow completely regular frames are a reflective (non-full) subcategory of completely regular frames

- denoted by $\mathscr{H}_\lambda L$ remove all dense λ-cozeros
- intersection of all the dense λ-cozero sublocales of L

$$\mathscr{H}_\lambda L = \bigcap_{\text{dense} a \in \lambda \text{Coz } L} \mathfrak{o}(a)$$

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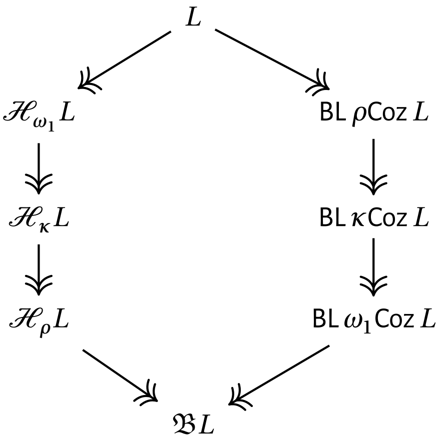
Hollowing sequence (descending):

$$L \supseteq \mathcal{H}_{\omega_1} L \supseteq \cdots \supseteq \mathcal{H}_\lambda L \supseteq \cdots \supseteq \mathcal{H}_\rho L = \mathfrak{B} L$$

Hollowing vs BL-towers



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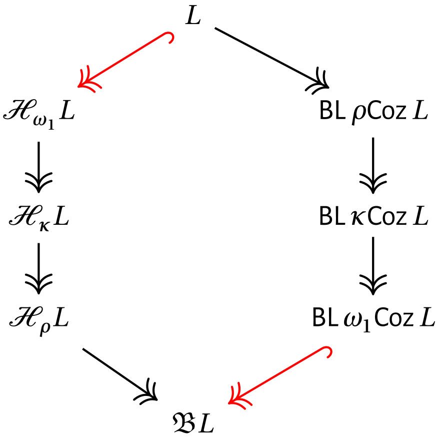

L is ω_1-hollow iff $L = \mathscr{H}_{\omega_1} L$ iff $\mathfrak{B}L = \mathrm{BL}\,\omega_1\mathrm{Coz}\,L$

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