Algebraic structures defined by the finite condensation on linear orders

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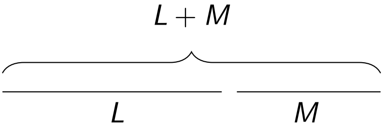
Linear orders and order types

- A relation $\preceq$ on a set X is a **linear order** if it is a partial order under which any two elements of X are comparable. That is, for all $x, y, z \in X$,
  1. $x \preceq x$ (reflexivity),
  2. $[(x \preceq y) \wedge (y \preceq x)] \implies x = y$ (antisymmetry), and
  3. $[(x \preceq y) \wedge (y \preceq z)] \implies x \preceq z$ (transitivity); and, in addition,
  4. $(x \preceq y) \vee (y \preceq x)$ (any two elements are comparable).
- If, in addition, every nonempty subset of X has a least element, X is a **well-ordering**.
- Notation for some common order types: $\omega = \text{o.t.}(\mathbb{N})$, $\omega^* = \text{o.t.}(\mathbb{N}^*)$ (the reverse ordering on $\mathbb{N}$), and $\zeta = \text{o.t.}(\mathbb{Z})$.

Addition of linear orders

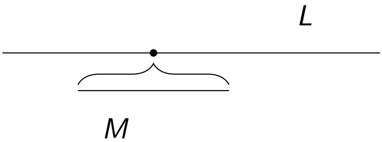
The sum $L + M$ is the linear order obtained by declaring all $l \in L$ to be less than all $m \in M$, while preserving the orders within L and M. That is, to form $L + M$, lay out a copy of L followed by a copy of M.



Note that addition of linear orders is not commutative.

(Lexicographic) multiplication of linear orders

The product LM is the linear order obtained by putting the lexicographic order on $L \times M$. That is, to form LM, one replaces each $l \in L$ with a copy of M.



- Multiplication of linear orders is not commutative either: $\omega 2 \cong \omega$, but $2\omega \cong \omega + \omega$.

The finite condensation

- For any linear order L, define $x \sim_F y$ iff there are only finitely many points of L between x and y.
- $\sim_F$ is a **condensation**: an equivalence relation whose equivalence classes are intervals of L (convex sets).
- $\sim_F$ is called the **finite condensation**.
- ${}^{\omega}\!/\!_{\sim_F} \cong 1$, because between any two natural numbers there are only finitely many points.
- ${}^{\omega^*}\!/\!_{\sim_F} \cong 1$,
- ${}^{\zeta}\!/\!_{\sim_F} \cong 1$,
- and ${}^{n}\!/\!_{\sim_F} \cong 1$ for any finite n.
- These are exactly the order types whose finite condensation is isomorphic to 1.

Multiplication mod the finite condensation

- Define an operation $\cdot_F$ on linear orders by $L \cdot_F M := \text{o.t.}(LM/{\sim_F})$ (the order type of the lexicographic product modulo the finite condensation).
- Set $R = \{1, \omega, \omega^*, \zeta\}$. We get the following multiplication table for $(R, \cdot_F)$:

$\cdot_F$	1	ω	ω^*	$\mathbb{Z}$
1	1	1	1	1
ω	1	ω	ω	ω
ω^*	1	ω^*	ω^*	ω^*
$\mathbb{Z}$	1	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$

Left rectangular band

- A *semigroup* is a set with an associative binary operation.
- A *band* is a semigroup in which every element is idempotent: for every x in the band, $x^2 = x$.
- A *left-rectangular band* is a band B such that $xyx = xy$ for all $x, y \in B$.

Proposition (B–S)

- $R = (\{1, \omega, \omega^*, \zeta\}, \cdot_F)$ *is a left rectangular band.*
- *Setting* $R^{\mathbf{ON}} = \{1, \omega\}$ *(the ordinal elements of* R*),* $(R^{\mathbf{ON}}, \cdot_F)$ *is a left rectangular sub-band.*

Endomorphisms

- An *endomorphism of* **ON** is a weakly order-preserving map f from **ON** to **ON**: $\alpha \leq \beta \implies f(\alpha) \leq f(\beta)$.
- Define $\phi^F_{\text{left}} : R^{\mathbf{ON}} \to \text{End}\,\mathbf{ON}$ by
  - $\phi^F_{\text{left}}(1)(\alpha) := 1 \cdot_F \alpha$
  - $\phi^F_{\text{left}}(\omega)(\alpha) := \omega \cdot_F \alpha$.
- Define $\phi^F_{\text{right}} : R^{\mathbf{ON}} \to \text{End}\,\mathbf{ON}$ by
  - $\phi^F_{\text{right}}(1)(\alpha) := \alpha \cdot_F 1$
  - $\phi^F_{\text{right}}(\omega)(\alpha) := \alpha \cdot_F \omega$.

Endomorphisms, cont'd.

Theorem (B–S)

Each of the maps $\phi^F_{\text{left}}(1)$, $\phi^F_{\text{left}}(\omega)$, $\phi^F_{\text{right}}(1)$, and $\phi^F_{\text{right}}(\omega)$ is an endomorphism of **ON**.

- $\phi^F_{\text{right}}(\omega)$ is the identity map from **ON** to **ON**.
- $\phi^F_{\text{left}}(1)$ and $\phi^F_{\text{right}}(1)$ are the map $\alpha \mapsto$ o.t.$(\alpha/\sim_F)$.

Endomorphisms, cont'd.

- ϕ^F_{left} and ϕ^F_{right} do not act like true representations in the sense of structure-preserving maps from the left rectangular band $R^{\mathbf{ON}}$ under $\cdot_F$ to the class End(**ON**) under composition.
- ϕ^F_{right} preserves the products $\omega \cdot_F 1$, $1 \cdot_F \omega$, and $\omega \cdot_F \omega$, but it does not preserve the product $1 \cdot_F 1$. (A similar situation holds for the map ϕ^F_{left}.)

Cantor normal form

Theorem (Cantor Normal Form)

Let α be an ordinal. Then α can be written in the form

$$n_1\omega^{\alpha_1} + \cdots + n_k\omega^{\alpha_k}$$

where $\alpha_1 > \alpha_2 > \cdots > \alpha_k$ are ordinals and where k and $n_1, \ldots, n_k$ are natural numbers (with $n_1 \neq 0$). Further, this decomposition is unique.

- If the exponents α_i are all finite, α has **finite degree**.

The Cantor normal form map

- We will denote by $\omega[\omega]^{\omega}_{CNF}$ the class of all Cantor normal forms of ordinals of finite degree, considered as formal sums.
- We define a map Φ by letting $\Phi(\alpha)$ be the Cantor normal form of α.
- When we restrict Φ to the ordinals of finite degree, we get the class map $\Phi : \{\alpha \in \mathbf{ON} : \deg(\alpha) < \omega\} \to \omega[\omega]^{\omega}_{CNF}$ sending an α of finite degree to its Cantor normal form.

Derivatives

Theorem (B–S)

Suppose α is an ordinal of finite degree with Cantor normal form $\Phi(\alpha) = a_n\omega^n + a_{n-1}\omega^{n-1} + \cdots + a_1\omega + a_0$, with $n > 0$. Then

$$1 \cdot_F \Phi(\alpha) = a_n\omega^{n-1} + a_{n-1}\omega^{n-2} + \cdots + a_1 + c_\alpha$$

where $c_\alpha = 0$ if $a_0 = 0$, and $c_\alpha = 1$ if $a_0 \neq 0$.

- Because of the resemblance of this map to an ordinary polynomial derivative, we give $\phi^F_{\text{left}}(1)$ the notation ∂_F.
- Notice that $\partial_F(\Phi(\alpha))$ is an element of $\omega[\omega]^\omega_{CNF}$.
- Observe that $\partial_F^{\deg(\alpha)+1}(\Phi(\alpha)) = 1$.

The finite condensation “derivative”

- We have the following commutative diagram for the finite condensation derivative ∂_F.

$$\begin{array}{ccc} \omega[\omega]^{\omega}_{CNF} & \xrightarrow{\partial_F} & \omega[\omega]^{\omega}_{CNF} \\ \Phi \uparrow & & \downarrow \Phi^{-1} \\ \{\alpha \in \mathbf{ON} : \deg(\alpha) < \omega\} & \overset{\partial_F^{\dagger}}{\dashrightarrow} & \{\alpha \in \mathbf{ON} : \deg(\alpha) < \omega\}). \end{array}$$

- Here, the derivative $\partial_F^{\dagger}$ mapping $\{\alpha \in \mathbf{ON} : \deg(\alpha) < \omega\}$ to $\{\alpha \in \mathbf{ON} : \deg(\alpha) < \omega\}$ is an induced derivative, arising from our definition of the operator ∂_F defined on $\omega[\omega]^{\omega}_{CNF}$.

Extending the finite condensation "derivative" onto ordinals

- Next, consider iterating $\partial_F^\dagger$. We have

$$\begin{aligned}\partial_F^\dagger \circ \partial_F^\dagger &= \partial_F^\dagger \circ (\Phi^{-1} \circ \partial_F \circ \Phi) \\ &= (\Phi^{-1} \circ \partial_F \circ \Phi) \circ (\Phi^{-1} \circ \partial_F \circ \Phi) \\ &= \Phi^{-1} \circ \partial_F \circ \mathrm{Id} \circ \partial_{\mathrm{F}} \circ \Phi \\ &= \Phi^{-1} \circ \partial_F^2 \circ \Phi.\end{aligned}$$

- Similarly, for $n < \omega$, we will have $(\partial_F^\dagger)^n = \Phi^{-1} \circ \partial_F^n \circ \Phi$.

Thank-you!

Thank-you to the organizers of BLAST for the support and the opportunity to speak today.

A couple of references

- Standard reference on linear orders:
  J. Rosenstein, *Linear Orderings*, Academic Press, 1982.
- Our paper associated with these slides:
  J. Brown and R. Suárez, Algebraic structures arising from the finite condensation on linear orders. (submitted; current version [v3] available on arXiv after 27 May 2025)