Simulation classes and aperiodicity

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BLAST 2025

University of Colorado, Boulder
May 19–23, 2025

Finite algebras viewed as finite-state automata

Remark

VanderWerf made the following philosophical claim in his PhD thesis "...what matters about finite algebras is what they can compute." It was VanderWerf's aim to order finite algebras corresponding to their computational power (as bottom-up tree automata). This led to the notion of a simulation class.

Definition

A collection of finite algebras is called a simulation class if it is closed under division, matrix powers, and wreath products. If $\mathcal{K}$ is a class of finite algebras, then the simulation class generated by $\mathcal{K}$, denoted by $\mathrm{Sim}(\mathcal{K})$, is the smallest simulation class containing $\mathcal{K}$.

Division

Definition

- Let $\mathbb{A}$ be an algebra. The clone of $\mathbb{A}$, denoted by $\mathrm{Clo}\mathbb{A}$, is the collection of all operations which may be obtained by composing all basic operations of $\mathbb{A}$ and the projection operations.
- We say that $\mathbb{B}$ is a reduct of $\mathbb{A}$ if $B = A$ and $\mathrm{Clo}\mathbb{B} \subseteq \mathrm{Clo}\mathbb{A}$.
- We say that $\mathbb{B}$ is a subreduct of $\mathbb{A}$ if it is a subalgebra of a reduct of $\mathbb{A}$.
- We say that $\mathbb{C}$ is a divisor of $\mathbb{A}$ if there is a subreduct $\mathbb{B}$ of $\mathbb{A}$ and a congruence θ of $\mathbb{B}$, such that $\mathbb{B}/\theta \cong \mathbb{C}$.
- We write $\mathbb{B} \leq \mathbb{A}$ if $\mathbb{B}$ is a subreduct of $\mathbb{A}$ and we write $\mathbb{C} \preceq \mathbb{A}$ if $\mathbb{C}$ is a divisor of $\mathbb{A}$.

Example

Let $\mathbb{L}_3 = \langle\{0,1,2\},\vee,\wedge\rangle$ denote the three-element lattice. The two-element join semilattice, $S = \langle\{0,1\},\vee\rangle$, is a subreduct of $\mathbb{L}_3$.

Matrix powers

Definition (Definition 13.2 Hobby and McKenzie)

Let $\mathbb{A}$ be an algebra and let k be a positive integer. The k^{th} matrix power of $\mathbb{A}$, denoted by $\mathbb{A}^{[k]}$, is defined as follows: The universe of $\mathbb{A}^{[k]}$ is A^k. For any $n \geq 0$ and $f_1, \ldots, f_k \in \mathrm{Clo}_{kn}\mathbb{A}$, there is an operation $f \in \mathrm{Clo}_n(\mathbb{A}^{[k]})$ defined by

$$f(\mathbf{a}_1, \ldots, \mathbf{a}_n) = (f_1(\mathbf{a}_1, \ldots, \mathbf{a}_n), \ldots, f_k(\mathbf{a}_1, \ldots, \mathbf{a}_n)),$$

where $\mathbf{a}_1, \ldots, \mathbf{a}_n \in A^k$.

Fact

- *$\mathrm{Con}\mathbb{A}$ and $\mathrm{Con}\mathbb{A}^{[k]}$ are naturally isomorphic.*
- *$\mathbb{A}$ and $\mathbb{A}^{[k]}$ have the same typeset.*

Matrix power example

Example

Let $\mathbb{L} = \langle \{0, 1\}, \vee, \wedge \rangle$ denote the two-element lattice. Consider the following operations in $\mathrm{Clo}(\mathbb{L}^{[2]})$:

$$
\begin{aligned}
j(\mathbf{x}, \mathbf{y}) &= (\mathbf{x}_1 \vee \mathbf{y}_1, \mathbf{x}_2 \wedge \mathbf{y}_2), \\
m(\mathbf{x}, \mathbf{y}) &= (\mathbf{x}_1 \wedge \mathbf{y}_1, \mathbf{x}_2 \vee \mathbf{y}_2), \\
\text{‘}(\mathbf{x}) &= (\pi_2(\mathbf{x}_1, \mathbf{x}_2), \pi_1(\mathbf{x}_1, \mathbf{x}_2)).
\end{aligned}
$$

Let $B = \{(0, 1), (1, 0)\}$. Claim: $\langle B, j(\mathbf{x}, \mathbf{y})|_B, m(\mathbf{x}, \mathbf{y})|_B, \text{‘}(\mathbf{x})|_B \rangle$ is a subreduct of $\mathbb{L}^{[2]}$ isomorphic to the two-element Boolean algebra.

Remark

The two-element Boolean algebra is a member of the simulation class generated by the two-element lattice.

Wreath products

Definition

Let $\mathbb{A}$ and $\mathbb{B}$ be finite algebras. The wreath product of $\mathbb{A}$ and $\mathbb{B}$, denoted by $\mathbb{A} \circ \mathbb{B}$, is defined as follows: The universe of $\mathbb{A} \circ \mathbb{B}$ is $A \times B$. For any $n \geq 0$, $g \in \mathrm{Clo}_n\mathbb{B}$, and $\alpha : B^n \to \mathrm{Clo}_n\mathbb{A}$, there is an operation $h \in \mathrm{Clo}_n(\mathbb{A} \circ \mathbb{B})$ defined by

$$h((a_1, b_1), \ldots, (a_n, b_n)) = (\alpha(b_1, \ldots, b_n)(a_1, \ldots, a_n), g(b_1, \ldots, b_n)),$$

where $(a_1, b_1), \ldots, (a_n, b_n) \in A \times B$.

Fact

- *Projection onto the second factor of a wreath product is a homomorphism. However, due to the asymmetry of the wreath product, projection onto the first factor need not be a homomorphism.*
- *Any finite algebra can be decomposed as a wreath product of simple algebras. This implies that a simulation class is determined by its simple members.*

Wreath product example

Example

Let $S = \langle \{0,1\}, \vee, 0, 1\rangle$ denote the two-element semilattice and let $\mathbb{Z}_3 = \langle \{0,1,2\}, +\rangle$ denote the three-element group. The following operation is a member of $\mathrm{Clo}_3(S \circ \mathbb{Z}_3)$.

$$f(\mathbf{x},\mathbf{y},\mathbf{z}) = \begin{cases} (\mathbf{x}_1 \vee \mathbf{y}_1 \vee \mathbf{z}_1, -(\mathbf{x}_2 + \mathbf{y}_2 + \mathbf{z}_2) \text{ mod } 3), & \text{if } (\mathbf{x}_2, \mathbf{y}_2, \mathbf{z}_2) \in D, \\ (1, -(\mathbf{x}_2 + \mathbf{y}_2 + \mathbf{z}_2) \text{ mod } 3), & \text{otherwise,} \end{cases}$$

where $D = \{(0,0,0), (1,1,1), (0,1,1), (1,0,1), (1,1,0)\}$.

Remark

Let θ denote the following equivalence relation on $\{0,1\} \times \{0,1,2\}$:

$$\{(0,0)\} \cup \{(0,1)\} \cup \{(0,2), (1,0), (1,1), (1,2)\}.$$

One can verify that θ is a congruence of $\langle \{0,1\} \times \{0,1,2\}, f(\mathbf{x},\mathbf{y},\mathbf{z})\rangle$. Let $\mathbb{A}_3 = \langle \{0,1\} \times \{0,1,2\}, f(\mathbf{x},\mathbf{y},\mathbf{z})\rangle / \theta$.

Simulation classes of interest

Definition

- An algebra $\mathbb{A}$ is primal if $Sim(\mathbb{A})$ is the class of all finite algebras.
- *Aprim* is the class of algebras $\mathbb{A}$ such that $Sim(\mathbb{A})$ is not the class of all finite algebras, such algebras are said to be aprimal.
- *StrSolv* is the class of all finite strongly solvable algebras.
- *Solv* is the class of all finite solvable algebras.
- *Aper* is the class of all algebras $\mathbb{A}$, such that $Sim(\mathbb{A})$ does not contain any finite group.

Example

- Any finite unary algebra is in *StrSolv*.
- Any finite Abelian group is in *Solv*.
- Any finite semilattice is in *Aper*.

A hierarchy of simulation classes

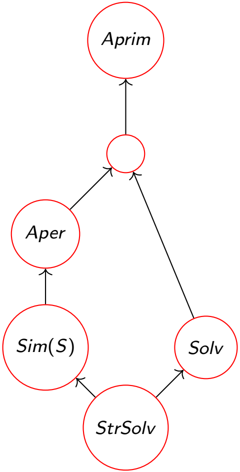

Figure: Some simulation classes of particular interest ordered by inclusion.

VanderWerf's decidability results

Definition

Let $\mathcal{C}$ be a simulation class. The membership problem for $\mathcal{C}$ asks if a given finite algebra $\mathbb{A}$ is a member of $\mathcal{C}$. The membership problem for $\mathcal{C}$ is decidable if there exists an algorithm which when given any finite algebra as input, outputs whether or not this algebra is a member of $\mathcal{C}$.

Theorem (VanderWerf 1994)

The membership problem is decidable for the following classes:

- *StrSolv*
- *Solv*
- *Aprim*

Exclusion classes

Definition

Let $\mathcal{K}$ be a class of finite algebras. The exclusion class of $\mathcal{K}$, denoted by $\mathrm{Excl}(\mathcal{K})$, is the class of all finite algebras that cannot simulate any algebra in $\mathcal{K}$.

Theorem (VanderWerf 1994)

- $\mathit{StrSolv} = \mathrm{Excl}\{S, \mathbb{Z}_p : p \text{ a prime.}\}$
- $\mathit{Solv} = \mathrm{Excl}\{S\}$
- $\mathit{Aprim} = \mathrm{Excl}\{\mathbb{B}\}$
- $\mathit{Aper} = \mathrm{Excl}\{\mathbb{Z}_p : p \text{ a prime.}\}$

A decision procedure for *Aprim*

Theorem (VanderWerf 1994)

The membership problem for Aprim is decidable.

Proof.

Let $\mathbb{A}$ be a finite algebra and let $\mathbb{B} = \langle \{0,1\}, \vee, \wedge, `(x) \rangle$ denote the two-element Boolean algebra.

- If $\mathbb{A} \notin Aprim$, then $\mathbb{B} \preceq \mathbb{A}^{[n_1]} \circ \cdots \circ \mathbb{A}^{[n_m]}$, for some $n_1, \ldots, n_m \geq 1$.
- If $\mathbb{B} \preceq \mathbb{A}^{[n_1]} \circ \cdots \circ \mathbb{A}^{[n_m]}$, then $\mathbb{B} \leq \mathbb{A}^{[n]}$ for some $n \geq 1$.
- If $\mathbb{B} \leq \mathbb{A}^{[n]}$ for some $n \geq 1$, then $\mathbb{B} \leq \mathbb{A}^{[m]}$ for $m \leq |A|^2$.

As $\mathbb{A}$ is finite, for any fixed integer m, there are finitely many subreducts of $\mathbb{A}^{[m]}$ with signature $(2,2,1)$. So, the above describes an algorithm which decides the membership problem for *Aprim*. □

The membership problem for *Aper*

Question (VanderWerf 1994)

Is the membership problem for Aper decidable?

Remark

- *If* $\mathbb{A} \notin Aper$*, then* $\mathbb{Z}_p \preceq \mathbb{A}^{[n_1]} \circ \cdots \circ \mathbb{A}^{[n_m]}$*, for some prime* p *and* $n_1, \ldots, n_m \geq 1$*.*
- *If* $\mathbb{Z}_p \preceq \mathbb{A}^{[n_1]} \circ \cdots \circ \mathbb{A}^{[n_m]}$*, then* $\mathbb{Z}_p \leq \mathbb{A}^{[n]}$ *for some* $n \geq 1$*.*
- *If* $\mathbb{Z}_p \leq \mathbb{A}^{[n]}$ *for some* $n \geq 1$*, then* $\mathbb{Z}_p \leq \mathbb{A}^{[m]}$ *for* $m \leq |A|^p$*.*

However, this only gives a decision procedure for a fixed prime p*. As there are infinitely many primes to check the above algorithm does not decide the membership problem for Aper.*

Separating $\mathrm{Sim}(S)$ and *Aper*

Question

VanderWerf demonstrated that Solv $= \mathrm{Sim}\{\mathbb{Z}_p : p \text{ a prime}\}$. *So one might ask the question: Is Aper* $= \mathrm{Sim}(S)$*?*

Theorem (VanderWerf 1994)

$\mathrm{Sim}(S)$ *and Aper are distinct simulation classes.*

Proof.

Let $\mathbb{A}_3$ be as previously defined. It can be demonstrated that $\mathbb{A}_3 \in Aper \setminus \mathrm{Sim}(S)$. $\square$

Conjecture (Bojańczyk 2004)

The membership problem for $\mathrm{Sim}(S)$ *is decidable.*

Results

Definition

Let p be an odd prime and let $\mathbb{A}_p = \langle \{0,1,2\}, f_p(x_1,\ldots,x_p)\rangle$, where

$$f(x_1,\ldots,x_p) = \begin{cases} 0, & \text{if } (x_1,\ldots,x_p) \in \{(0,\ldots,0),(1,\ldots,1)\}, \\ 1, & \text{if } (x_1,\ldots,x_p) \in \{(0,1,\ldots,1),\ldots,(1,\ldots,1,0)\}, \\ 2, & \text{otherwise} \end{cases}$$

Theorem

- $\mathbb{A}_p \in \mathrm{Sim}(S,\mathbb{Z}_p)$ *[2021].*
- $\mathbb{A}_p \in$ *Aper [2021].*
- $\mathbb{A}_p \notin$ *Sim*(S) *[2021].*
- $\mathbb{A}_p$ *is not a member of Sim*$\{\mathbb{A}_q : q$ *an odd prime,* $q \neq p\}$ *[2023].*

Corollary

There are continuum many simulation classes contained in Aper.

What is known about *Aper*?

Theorem (VanderWerf 1994)

Let $\mathbb{A}$ be a finite algebra. The following are equivalent.

1. $\mathbb{A} \in Aper$.
2. $\mathbb{A} \in Aprim$ *and there is no subreduct* $\mathbb{B}$ *of* $\mathbb{A}^{[k]}$ *that has a type* **2** *subtrace for any* $k \geq 1$.

Theorem (VanderWerf 1994)

Let $\mathbb{A}$ be a finite simple algebra of type **5**. *If the trace order of $\mathbb{A}$ has a unique minimal element, then $\mathbb{A} \in Aper$.*

The end

Thank you!