A Characterization of Finitely Based Abelian Mal'cev Varieties

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February 6, 2025

Algebras and Varieties

An *algebra* $\mathbf{A}$ is a set with function symbols. These symbols are interpreted as operations on the set.

Example

- the group structure $\langle \mathbb{Z}, +, -, 0 \rangle$

The set of function symbols is the algebra's *signature*.

A *variety* $\mathcal{V}$ is a class of algebras with the same signature that is defined by a set of equations Σ. Then Σ is a *basis* for $\mathcal{V}$.

Example

The variety of groups is the variety with signature $\langle +, -, 0 \rangle$ satisfying the following

G1 $(x + y) + z \approx x + (y + z)$

G2 $x + 0 \approx 0 + x \approx x$

G3 $x + (-x) \approx (-x) + x \approx 0$

Finitely based varieties

A variety is *finitely based* if its basis of equations Σ is finite in size.

Example

- The variety of groups is finitely based.
- The variety of lattices is finitely based.

Examples of Finite Basis Results

Example

- (A. Yu. Ol'shanskii, 1970) There exists varieties of groups which are not finitely based.
- (Lyndon, 1952) The variety of k-nilpotent groups is finitely based for every $k \in \mathbb{N}$.

Mal'cev algebras

A variety $\mathcal{V}$ is Mal'cev if there exists a ternary term operation $m(x, y, z)$, called a *Mal'cev term*, such that

$$\mathcal{V} \models m(x, x, y) \approx y \approx m(y, x, x).$$

Example

Any expansion of additive groups has Mal'cev term $m(x, y, z) = x - y + z$.

Abelian Algebras

Theorem (H.P. Gumm, 1980)

A variety with Mal'cev term $m(x, y, z)$ is abelian if and only if for every basic operation f we have

$$\mathcal{V} \models m(f(\overline{x}), f(\overline{y}), f(\overline{z})) \approx f(m(x_1, y_1, z_1), \ldots, m(x_k, y_k, z_k)),$$

where f is assumed to have arity k and $\overline{x}$ is a k-tuple.

Example

$\langle \mathbb{Z}, x - y + z, 3x + 4y + 3 \rangle$ is an abelian algebra

Free Algebras

For a variety $\mathcal{V}$ and variables $x_1, \ldots, x_n$, we let $\mathbf{F}_{\mathcal{V}}(x_1, \ldots, x_n)$ denote the *free algebra over* $x_1, \ldots, x_n$ *in* $\mathcal{V}$.
We let $F_{\mathcal{V}}^{id}(x_1, \ldots, x_n) = \{t \in F_{\mathcal{V}}(x_1, \ldots, x_n) \mid t(z, \ldots, z) = z\}$

Module Structure

Theorem (c.f. R. Freese, R. McKenzie, 1987)

Let $\mathcal{V}$ be an abelian variety with Mal'cev term m. For $s, t \in F_{\mathcal{V}}(x, z)$ define

$$\begin{aligned} s + t &:= m(s(x,z), z, t(x,z)), \\ s \cdot t &:= s(t(x,z), z) \\ -s &:= m(z, s(x,z), z). \end{aligned}$$

1. $\mathbf{R}_{\mathcal{V}} := \langle F^{id}_{\mathcal{V}}(x, z), +, -, \cdot \rangle$ is a ring with identity x and zero z.
2. $\mathbf{M}_{\mathcal{V}} := \langle F_{\mathcal{V}}(z), +, -, R_{\mathcal{V}} \rangle$ is an $\mathbf{R}_{\mathcal{V}}$-module with zero z.
3. $\langle F_{\mathcal{V}}(x, z), +, -, R_{\mathcal{V}} \rangle \cong R_{\mathcal{V}} \oplus \mathbf{M}_{\mathcal{V}}$ as $\mathbf{R}_{\mathcal{V}}$-modules

Characterization Theorem

Theorem (M. Muro, 2024)

Let $\mathcal{V}$ be an abelian Mal'cev variety of finite signature. Then $\mathcal{V}$ is finitely based if and only if

1. the ring $\mathbf{R}_{\mathcal{V}}$ of binary idempotent terms is finitely presented,
2. and the $\mathbf{R}_{\mathcal{V}}$-module $\mathbf{M}_{\mathcal{V}}$ of unary terms is finitely presented.

Proof of Characterization Theorem

Lemma (c.f. Szendrei, 1980)

Every abelian Mal'cev variety $\mathcal{V}$ of finite signature is equivalent to a subvariety $\mathcal{W}$ of $\mathcal{U}$ in the signature $\{u_1, \ldots, u_\ell, r_1, \ldots, r_n, m\}$ defined by

- $m(x, y, y) \approx x \approx m(y, y, x)$.
- $r_i(z, z) \approx z$ for all $1 \leq i \leq n$.
- $m\big(u_i(x), u_i(y), u_i(z)\big) \approx u_i\big(m(x, y, z)\big)$ for all $1 \leq i \leq \ell$.
- $m\big(r_i(x_1, x_2), r_i(y_1, y_2), r_i(z_1, z_2)\big) \approx$ $r_i\big(m(x_1, y_1, z_1), m(x_2, y_2, z_2)\big)$ for all $1 \leq i \leq n$.
- $m\big(m(x_1, x_2, x_3), m(y_1, y_2, y_3), m(z_1, z_2, z_3)\big) \approx$ $m\big(m(x_1, y_1, z_1), m(x_2, y_2, z_2), m(x_3, y_3, z_3)\big)$.

$\mathcal{U}$

Lemma

The variety $\mathcal{U}$ has the following properties

- The variety $\mathcal{U}$ is finitely based.
- The ring $\mathbf{R}_{\mathcal{U}}$ is free over $r_1, \ldots, r_n$.
- The $\mathbf{R}_{\mathcal{U}}$-module $\mathbf{M}_{\mathcal{U}}$ is free over $u_1, \ldots, u_\ell$.
- The variety $\mathcal{W}$ is a subvariety of $\mathcal{U}$.
- $\mathbf{R}_{\mathcal{W}}$ is a quotient of $\mathbf{R}_{\mathcal{U}}$ and $\mathbf{M}_{\mathcal{W}}$ is a quotient of $\mathbf{M}_{\mathcal{U}}$.

Fully Invariant Congruences

Lemma

A congruence θ on $\mathbb{F}_{\mathcal{U}}(x, z)$ is fully invariant if and only if

- $I = z/\theta \cap R_{\mathcal{U}}$ is an ideal of $\mathbf{R}_{\mathcal{U}}$,
- $N = z/\theta \cap M_{\mathcal{U}}$ is an $\mathbf{R}_{\mathcal{U}}$-submodule of $\mathbf{M}_{\mathcal{U}}$,
- $z/\theta = I + N$,
- $IM_{\mathcal{U}} \subseteq N$,
- and $f(x) - f(z) - x \in I$ for all $f(z) \in N$, f a unary term of $\mathcal{U}$.

Finitely Generated Fully Invariant Congruences

Lemma

A fully invariant congruence θ on $\mathbb{F}_{\mathcal{U}}(x, z)$ is finitely generated if and only if

1. the ideal $I = z/\theta \cap R_{\mathcal{U}}$ of $\mathbf{R}_{\mathcal{U}}$ is finitely generated and
2. for $N = z/\theta \cap M_{\mathcal{U}}$ the $\mathbf{R}_{\mathcal{U}}$-module $N/IM_{\mathcal{U}}$ is finitely generated.

We have all the parts to prove that $\mathcal{W}$ is finitely based if and only if $\mathbf{R}_{\mathcal{W}}$ and $\mathbf{M}_{\mathcal{W}}$ are finitely presented.
Let $\phi\colon \mathbf{F}_{\mathcal{U}}(x,\ldots) \to \mathbf{F}_{\mathcal{W}}(x,\ldots),\ t^{\mathcal{U}} \mapsto t^{\mathcal{W}}$ and let $\theta = \ker\phi$.

- If $\mathcal{W}$ is finitely based, then θ is finitely generated as a fully invariant congruence of $\mathbf{F}_{\mathcal{U}}(x_1,\ldots)$.
- This means that θ uniquely determines a finitely generated ideal I of $\mathbf{R}_{\mathcal{U}}$ and finitely generated submodule N of $\mathbf{M}_{\mathcal{U}}$.
- We have $\mathbf{R}_{\mathcal{W}} \cong \mathbf{R}_{\mathcal{U}}/I$ and $\mathbf{M}_{\mathcal{W}} \cong \mathbf{M}_{\mathcal{U}}/N$.
- Since $\mathbf{R}_{\mathcal{U}}$ and $\mathbf{M}_{\mathcal{U}}$ are free with finitely many generators, that means $\mathbf{R}_{\mathcal{W}}$ and $\mathbf{M}_{\mathcal{W}}$ are finitely presented.

The backwards direction is similar.

References

[M. Muro, 2024] Characterizing Finitely Based Abelian Mal'cev Algebras - https://arxiv.org/abs/2411.17004