Local compactness does not always imply spatiality

Sebastian D. Melzer

New Mexico State University

May 19, 2025 – BLAST, Boulder

Joint work with Guram Bezhanishvili, Ranjitha Raviprakash, and Anna L. Suarez

Pointfree topology

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The motivating example is that for each topological space X, the lattice of open sets $\Omega(X)$ forms a frame. Frames of this form are called spatial.

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Although it became central in modal logic, it was largely overlooked in pointfree topology, but recent work reintroduced McKinsey–Tarski (MT) algebras into the pointfree study of spaces.

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Moreover, for each frame L there exists an MT-algebra M such that $\mathcal{O}(M) = L$ (the Funayama envelope $\mathscr{F}(L)$ of L). In this sense, the MT-algebra setting generalizes that of frames.

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These definitions are also compatible with frame theory: under mild assumptions (e.g., T_1), M is T_i iff $\mathcal{O}(M)$ is T_i for $i = 3, 3\frac{1}{2}, 4$.

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If M is T_2, then $\mathcal{O}(M)$ is T_2. The converse was an open question. As it turns out, the main counterexample in this talk also answers this in the negative.

Kubiak's comment

In a review of [GR, 2023], the following observation was made:

> "A coherent system of separation axioms [...] for MT-algebras is given by G. Nöbeling in his pioneering book[1] [...]. It is rather immediate (except possibly for $i = 2$) that an MT-algebra B is Nöbeling-T_i if and only if it is T_i ..."
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Despite being mentioned briefly by Johnstone (1982) as an early example of pointfree topology, this remark points to a largely forgotten chapter in the history of pointfree topology. Nöbeling appears to be among the last to develop pointfree topology in the powerset-inspired direction, just before frame theory took over.

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His work includes separation axioms, local compactness, and spatiality results in this setting.

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Comparing separation axioms

Sparing you details, Kubiak is correct that the definitions (except for T_2) are equivalent.

Lemma

Let M be an MT-algebra. For $i = 1, 3, 3\frac{1}{2}, 4$, M is T_i iff M is Nöbeling-T_i.

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For T_2, however, the definitions diverge.

The classical T_2 condition

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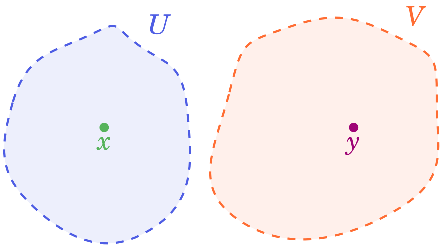

The Nöbeling T_2 condition

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$$a \neq 0,\ b \neq 0, \text{ and } a \wedge b = 0 \implies \exists u, v \in \mathcal{O}(M) \text{ with } u \wedge a \neq 0,\ v \wedge b \neq 0, \text{ and } u \wedge v = 0.$$

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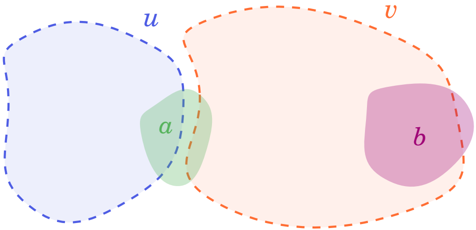

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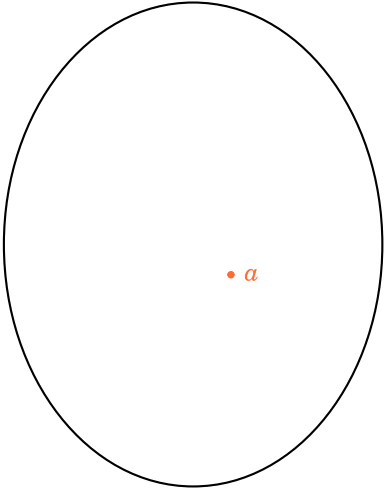

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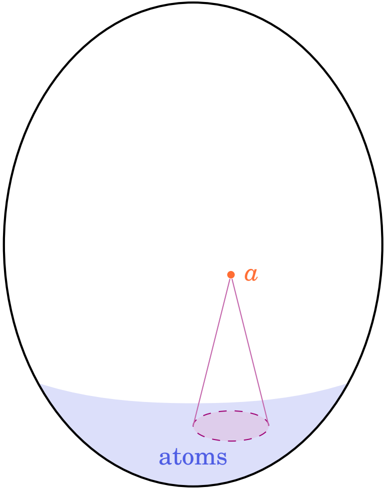

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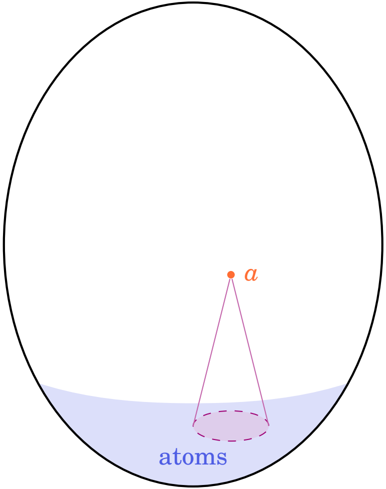

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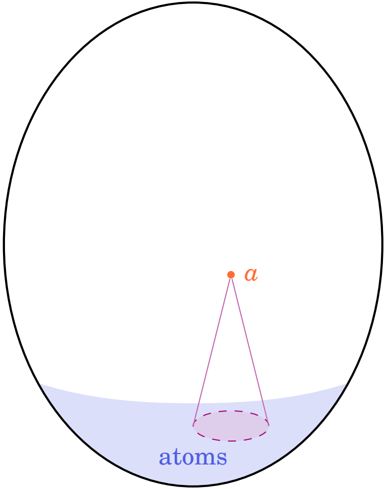

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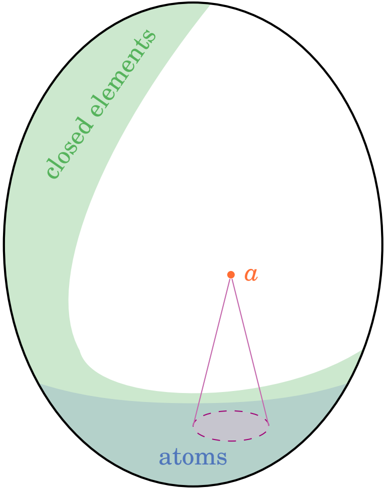

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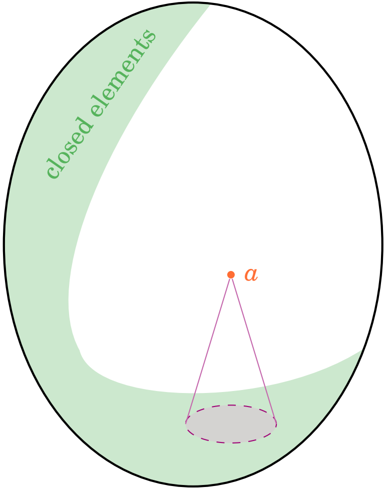

Comparing Hausdorff

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Thus, in the MT setting, T_2 can be expressed as: every element is a join of elements of the form

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Theorem

1. *If M is T_2, then M is Nöbeling-T_2.*
2. *There exist Nöbeling-T_2 MT-algebras that are not T_1 (and hence not T_2).*

Example

Let B be a complete atomless Boolean algebra. Put $M = B \times B$ with $\Box(a,b) = (a \wedge b, b)$.

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The fact that Nöbeling-locally compact and locally compact are very different is unsurprising.

Let X be a topological space. Most commonly X is called **locally compact** if every point x of X has a compact neighbourhood, i.e., there exists an open set U and a compact set K, such that $x \in U \subseteq K$.

There are other common definitions: They are all **equivalent if *X* is a Hausdorff space** (or preregular). But they are **not equivalent** in general:

1. every point of X has a compact neighbourhood.
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2″. every point of X has a local base of relatively compact neighbourhoods.

3. every point of X has a local base of compact neighbourhoods.
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5. X is Hausdorff and satisfies any (or equivalently, all) of the previous conditions.

Logical relations among the conditions:[2]

- Each condition implies (1).
- Conditions (2), (2′), (2″) are equivalent.
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In short, they generalize different properties.

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There is a connection between T_1 MT-algebras and subfit frames, which allows one to derive ISBELL's Spatiality Theorem from NÖBELING's result.

Corollary (Isbell, 1972)

Compact subfit frames are spatial.

Nöbeling-locally compact T_1 algebras are spatial

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T_{1/2} \arrow[d, dash] \arrow[drr, dash] & & \text{Sober} \arrow[d, dash] \\
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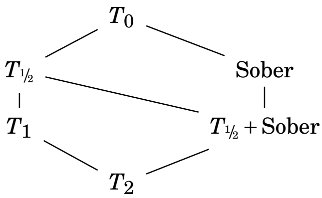
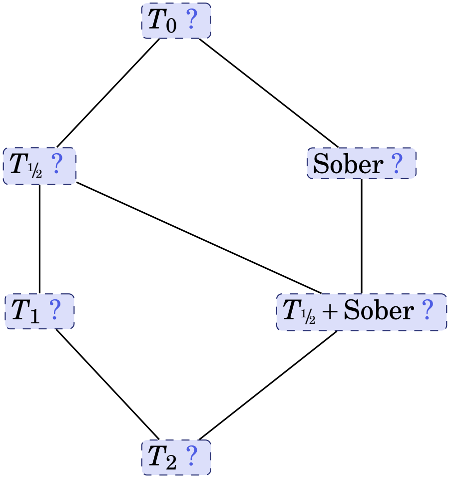


This leads to a natural question: how much separation is actually needed to make Nöbeling-locally compact MT-algebras spatial?

Which Nöbeling-locally compact MT-algebras are spatial?



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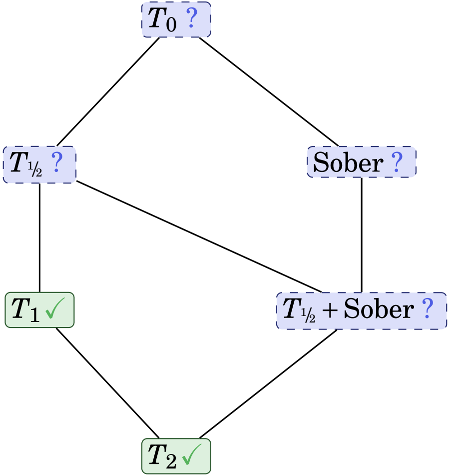

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Example

Let L be a complete atomless Boolean algebra with an extra top element adjoined. Then $\mathcal{F}(L)$ is compact, sober, and $T_{1/2}$, but has only one atom.

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This shows that Nöbeling-local compactness does not provide enough local information to ensure spatiality.

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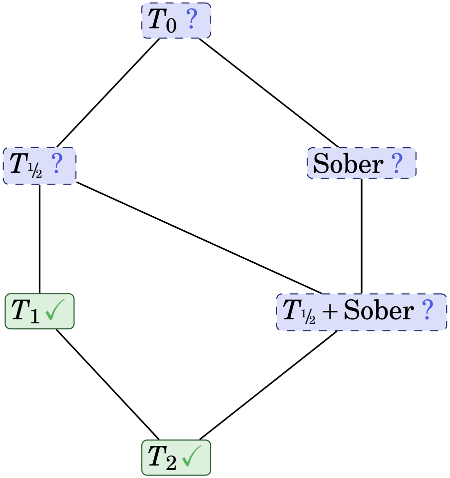

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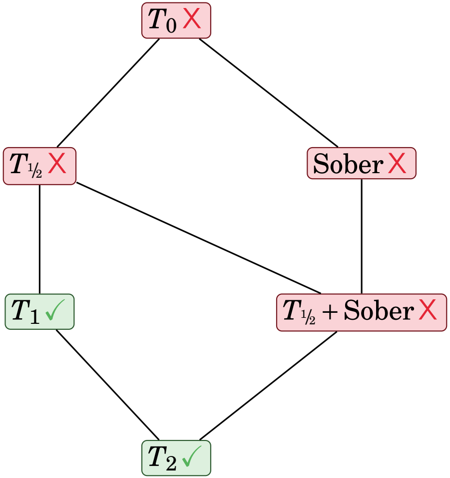

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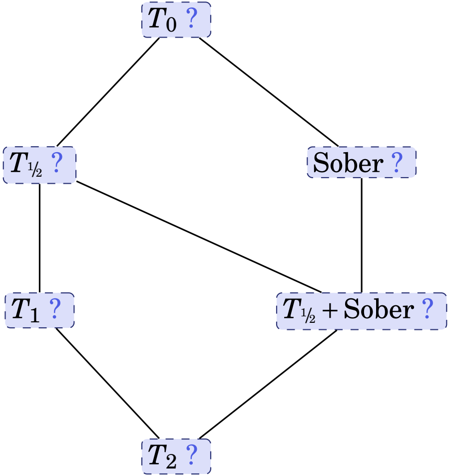
We will see that this changes when working with locally compact MT-algebras.

Which locally compact MT-algebras are spatial?



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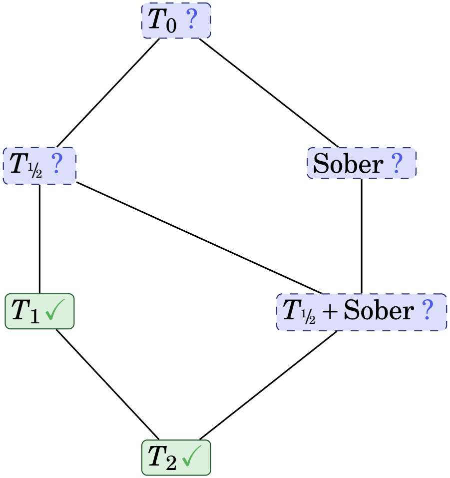

The $T_{1/2}$ + sober case

When working with sober $T_{1/2}$ MT-algebras, spatiality can be recovered from the frame:

Theorem (Guram–Ranjitha, 2023)

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Combining the two gives:

Corollary

Locally compact sober $T_{1/2}$ MT-algebras are spatial.

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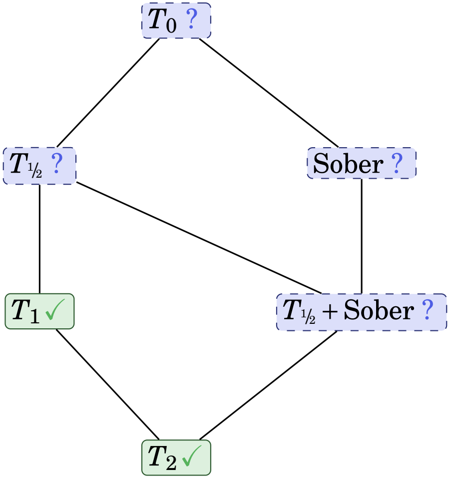

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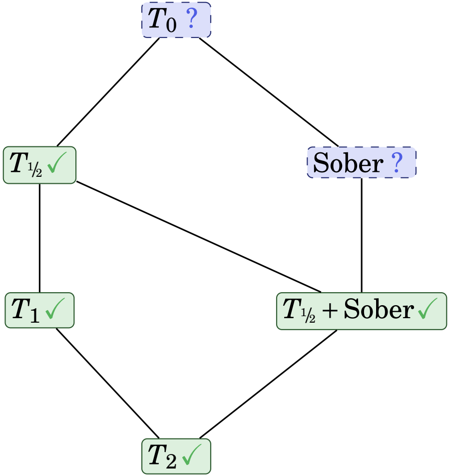

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Instead, we turn to Raney extensions, as studied by ANNA.

Roughly speaking, just as frames correspond to lattices of open sets, Raney extensions correspond to lattices of saturated sets.

Raney extensions and T_0

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We also have:

Lemma

Let C be a Raney extension.

1. *C is spatial iff $\mathcal{F}(C)$ is spatial.*
2. *C is sober iff $\mathcal{F}(C)$ is sober.*

The main example

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In this case, C is sober but not spatial. Therefore, $\mathscr{F}(C)$ is sober and not spatial.

Moreover, since $\Omega(\mathbb{R})$ is continuous and $\mathscr{F}(C)$ is sober, $\mathscr{F}(C)$ is locally compact.

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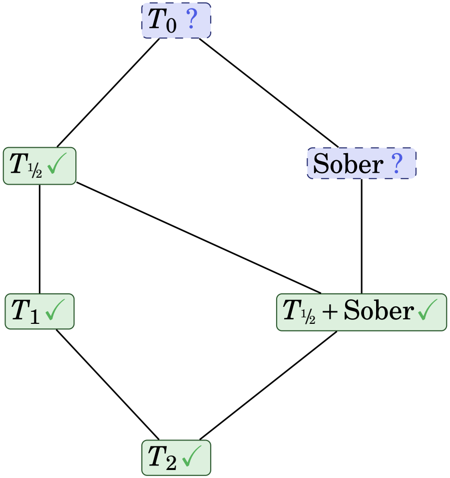
Moreover, since $\Omega(\mathbb{R})$ is continuous and $\mathscr{F}(C)$ is sober, $\mathscr{F}(C)$ is locally compact.

Theorem

There exist locally compact sober MT-algebras that are not spatial.

Which locally compact MT-algebras are spatial?



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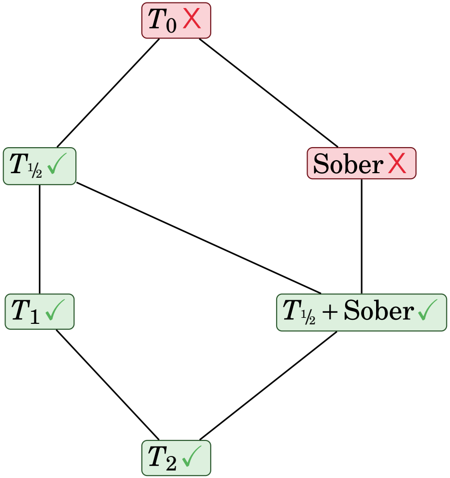

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Local compactness does not always imply spatiality


Thank you!