A Baer-like criterion for relative injective modules via model theory

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Background

- Abstract elementary classes constitute a model theoretic framework to study classes of structures.
- They were introduced by Saharon Shelah in the seventies.
- The abstract theory has developed rapidly in the last 25 years.
- Recently there has been a push to find interactions and applications to algebra
- Interaction between ring theory and superstability
- Solution to a problem of Fuchs
- Baer-like criterion for relative injective modules

Outline

1. **Basic notions**
2. Dividing lines
3. Applications (joint work with J. Rosický)
4. Summary and future work

Basic notions: Module theory

R is an associative ring with unity.

R-Module

An R-module is a "vector space" over the ring R.

Some key examples:

- $\mathbb{R}$-modules are vectors spaces over $\mathbb{R}$.
- $\mathbb{Z}$-modules are abelian groups.

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Strong submodules (Prüfer 1930s)

- $A \leq_{\text{RD}} B$ if and only if divisibility is preserved between A and B.
- $A \leq_p B$ if and only if existence of solutions to systems of linear equations are preserved between A and B.

For $\mathbb{Z}$-modules: $\leq_{\text{RD}} = \leq_p$.

Examples

$M \leq_p M \oplus N$, $t(G) \leq_p G$, $\mathbb{Z} \not\leq_p \mathbb{Q}$.

Basic notions: Abstract elementary class (Shelah 1970s)

An *abstract elementary class* is a pair $\mathbf{K} = (K, \leq_{\mathbf{K}})$ where K is class of mathematical structures and $\leq_{\mathbf{K}}$ is a partial order on K.

Key axioms

1. $\mathbf{K}$ is closed under isomorphisms.
2. If $M \leq_{\mathbf{K}} N$, then M is a substructure of N.
3. $\mathbf{K}$ is closed under unions of increasing chains.
4. Every model in K can be decomposed into *small* submodels in K.

In this talk:

- $K \subseteq$ Ab, R-Mod or $K \subseteq S$-Act
- $\leq_{\mathbf{K}}$ will be $\leq$ or $\leq_p$ or $\leq_{RD}$.

Basic notions: Examples

- $(R\text{-Mod}, \leq)$, $(R\text{-Mod}, \leq_p)$ and $(R\text{-Mod}, \leq_{RD})$

- $(R\text{-Flat}, \leq_p)$.
  For $R = \mathbb{Z}$, G is R-Flat if every element of G has infinite order.

- $(\aleph_1\text{-free}, \leq_p)$.
  G is an $\aleph_1$-free group if every countable subgroup is free.

- $(S\text{-Act}, \leq)$, $(S\text{-Act}, \leq_p)$ for S a monoid

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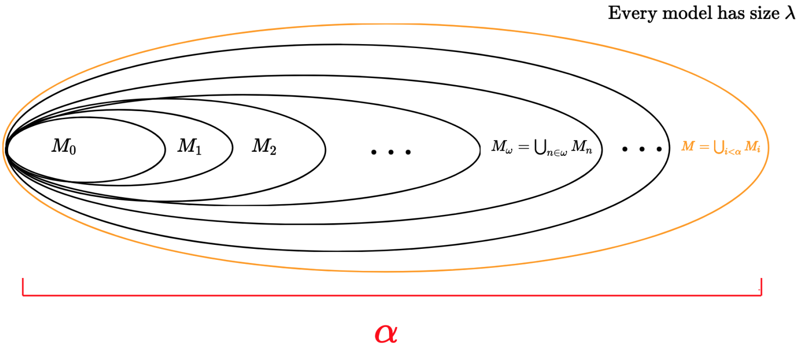
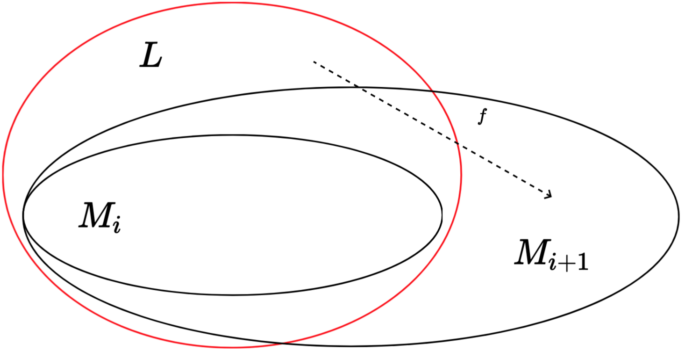
Dividing lines: Limit models (Kolman-Shelah 1990s)

M is a λ-limit model if there is $\alpha < \lambda^+$ limit ordinal and there is $\{M_i : i < \alpha\} \subseteq K$:



Dividing lines: Limit models

And for every $i < \alpha$, M_{i+1} is universal over M_i:



In this case, M is a (λ, α)-limit model.

Dividing lines: Stability

Stable (Shelah 1990s)

- $\mathbf{K}$ is λ-stable if $\mathbf{K}$ has a λ-limit model.
- $\mathbf{K}$ is stable if there is a λ such that $\mathbf{K}$ is λ-stable.

Dividing lines: Stability

Stable (Shelah 1990s)

- **K** is λ-stable if **K** has a λ-limit model.
- **K** is stable if there is a λ such that **K** is λ-stable.

Theorem (Fisher-Bauer 1970s)

If T is a complete first-order theory of modules, then $(Mod(T), \leq_p)$ is stable.

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Question (M. 2020)

If $(K, \leq_p)$ is an **abstract elementary class of modules**, is $(K, \leq_p)$ stable? Is it true if $R = \mathbb{Z}$? Under what conditions on R is it true?

Dividing lines: Superstability

Recall: In the definition of λ-limit model there is a parameter $\alpha < \lambda^+$.

Uniqueness of limit models

K has *uniqueness of limit models of cardinality* λ if **K** has a single λ-limit model up to isomorphisms.

Superstability (Shelah, Grossberg-Vasey)

K is *superstable* if and only if **K** has uniqueness of limit models in *all* large cardinals.

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K is *superstable* if and only if **K** has uniqueness of limit models in *all* large cardinals.

If **K** is superstable, then **K** is stable.

T is a complete first-order theory: $(Mod(T), \preceq)$

$(Mod(T), \preceq)$ is superstable if and only if T is superstable.

Dividing lines: Superstability

Question

If **K** is an abstract elementary class of algebraic objects, under what conditions is **K** superstable? Is there an algebraic reason why this happens?

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Noetherian rings (1921): Every ideal is finitely generated.

Characterizing noetherian rings (M.)

Let R be an associative ring with unity. The following are equivalent.

1. R is left noetherian.
2. The class of absolutely pure left R-modules with pure embeddings is superstable
3. The class of left R-modules with embeddings is superstable.

Similar results for: pure semisimple rings and perfect rings.

Dividing lines: Superstability

Weakly noetherian monoids: Every ideal is finitely generated.

Characterizing weakly noetherian monoids (M.-Rosický)

Let S be a monoid. The following are equivalent.

1. S is left weakly noetherian.
2. The class of left S-acts with embeddings is superstable.

Dividing lines: Superstability

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Question

Can the study of stability and superstability for other varieties give interesting information?

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3. **Applications (joint work with J. Rosický)**
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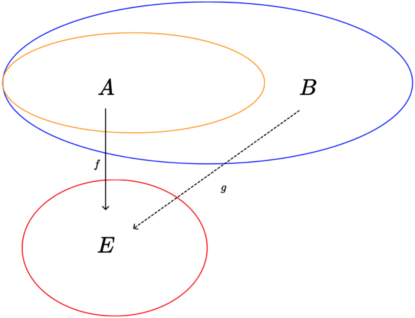
Applications: Injective modules

Injective module

E is injective if for every modules A and B such that $A \leq B$ and every homomorphism $f : A \to E$, there is a homomorphism $g : B \to E$ such that $g \restriction A = f$



Injective $\mathbb{Z}$-modules are divisible abelian groups: $\mathbb{Q}$, $\mathbb{Z}(p^\infty)$.

Applications: Hypothesis

Hypothesis

Let $(K, \mathcal{M})$ be a pair such that:

1. K is a class of R-modules for a fixed ring R,
2. $\mathcal{M}$ is either the class of embeddings, RD-embeddings or pure embeddings, and
3. K is closed under:
    1. direct sums,
    2. $\mathcal{M}$-submodules, i.e., if $A \in K$ and $B \leq_{\mathcal{M}} A$, then $B \in K$, and
    3. $\mathcal{M}$-epimorphic images, i.e., if $f : A \to B$ is a $\mathcal{M}$-epimorphism and $A \in K$, then $B \in K$.

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   3. $\mathcal{M}$-epimorphic images, i.e., if $f : A \to B$ is a $\mathcal{M}$-epimorphism and $A \in K$, then $B \in K$.

Always an AEC

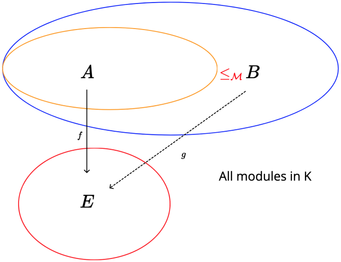
$K_{\mathcal{M}} = (K, \leq_M)$ is an AEC.

$(R\text{-Modules}, \text{Emb}/\text{RD}/\text{Pure})$
$(R\text{-Flat}, \text{Pure})$, $(R\text{-Tor}, \text{Pure})$

Applications: Relative injective modules

Relative injective module

E is $K_{\mathcal{M}}$-injective module if $E \in K$ and for every $A, B \in K$ such that $A \leq_{\mathcal{M}} B$ and every homomorphism $f : A \to E$, there is a homomorphism $g : B \to E$ such that $g \upharpoonright A = f$



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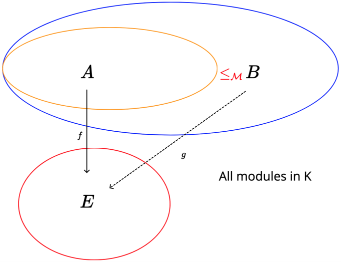


- When (R-Modules, Emb) we get injective modules.
- When (R-Modules, Pure/RD) we get pure/RD injective modules
- When (R-Flat, Pure) we get cotorsion flat modules

Applications: Baer criterion

Baer's Criterion (1940)

E is injective if and only if for every ideal I and every homomorphism $f : I \to E$, there is a homomorphism $g : R \to E$ such that $g \upharpoonright I = f$.

Applications: Baer criterion

Baer's Criterion (1940)

E is injective if and only if for every ideal I and every homomorphism $f : I \to E$, there is a homomorphism $g : R \to E$ such that $g \restriction I = f$.

Question

Can one extend Baer Criterion to relative injective modules?

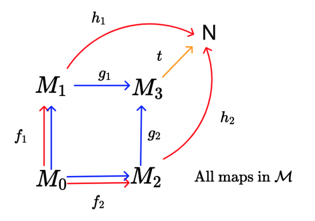
One cannot restrict to ideals such that $I \leq_{\mathcal{M}} R$.
$(Ab, \leq_p)$: If $I \leq_p \mathbb{Z}$, then $I = \mathbb{Z}$ or $\{0\}$.

Applications: Independence relations

Extending linear independence and non-forking to arbitrary categories.

Independence relation (Lieberman-Rosický-Vasey 2019)

An independence relation on $(K, \mathcal{M})$ is a set $\mathop{\smile}\limits^{\vert}$ of commutative squares in $\mathcal{M}$ such that for any commutative diagram:



we have that $(f_1, f_2, g_1, g_2) \in \mathop{\smile}\limits^{\vert}$ if and only if $(f_1, f_2, h_1, h_2) \in \mathop{\smile}\limits^{\vert}$.
Elements of $\mathop{\smile}\limits^{\vert}$ are called independent squares.

Applications: Independence relations

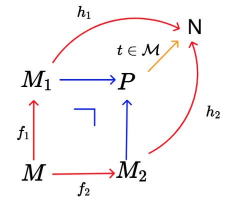
Stable independence relation (Lieberman-Rosický-Vasey 2019)

An independence relation $\mathop{\smile}\limits^{\mid}$ is a stable independence relation if:

- Existence: every span $M_1 \leftarrow M_0 \rightarrow M_2$ can be completed to an independent square.
- Uniqueness: there is a unique independent square for every span (up to equivalence).
- Symmetry: the *ears* of an independent squares can be switched.
- Transitivity: composition of independent squares is an independent square.
- Local character: given $M_1 \rightarrow N \leftarrow M_2$ there is a *small* module that completes the diagram to an independent square.
- Witness property: Checking if an independent square is an independent square can be done by checking for *small* sets.

An independence relation

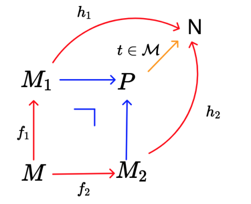
$(f_1, f_2, h_1, h_2) \in \mathop{\smile}\limits^{|}$ if and only if all the arrows of the outer square are $\mathcal{M}$-embeddings, (P, g_1, g_2) is the pushout of (M, f_1, f_2), and the unique map $t : P \to N$ is a $\mathcal{M}$-embedding:



Applications: Independence relations

An independence relation

$(f_1, f_2, h_1, h_2) \in \mathop{\smile\hskip-0.9em\vert\hskip0.6em}$ if and only if all the arrows of the outer square are $\mathcal{M}$-embeddings, (P, g_1, g_2) is the pushout of (M, f_1, f_2), and the unique map $t : P \to N$ is a $\mathcal{M}$-embedding:



Theorem (M.-Rosický)

$\mathop{\smile\hskip-0.9em\vert\hskip0.6em}$ as defined above is a stable independence relation with the $(< \aleph_0)$-witness property. In particular, $K_{\mathcal{M}}$ is stable.

Applications: Baer-like criterion

Baer-like criterion for relative injective modules (M.-Rosický)

Assume $E \in K$. E is a $K_{\mathcal{M}}$-injective module if and only if if for every $A, B \in K$ such that $A \leq_{\mathcal{M}} B$ and $\|A\|, \|B\| \leq \text{card}(R) + \aleph_0$ and for every $f : A \to E$ a R-homomorphism there is $g : B \to E$ a R-homomorphism such that $g \restriction A = f$.

Applications: Baer-like criterion

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Assume $E \in K$. E is a $K_{\mathcal{M}}$-injective module if and only if if for every $A, B \in K$ such that $A \leq_{\mathcal{M}} B$ and $\|A\|, \|B\| \leq \mathrm{card}(R) + \aleph_0$ and for every $f : A \to E$ a R-homomorphism there is $g : B \to E$ a R-homomorphism such that $g \restriction A = f$.

Proof. By induction on $\|B\|$. Suppose $\|B\| > \mathrm{card}(R) + \aleph_0$

- Decompose $A \leq_{\mathcal{M}} B$ into small $\{A_i : i < \lambda\}$, $\{B_i : i < \lambda\}$ which are independent of each other, i.e., if $i < j$, then $B_i \underset{A_i}{\overset{B_j}{\downarrow}} A_j$ (intuition $B_i \cap A_j = A_i$).
- Build $\{g_i : i < \lambda\}$ such that for every $i < \lambda$, $f \restriction A_i \subseteq g_i$ and $g_i : B_i \to E$.
- $i + 1$: $dom(g_i) \cap dom(f \restriction A_{i+1}) = A_i$ and g_i agrees with f up to that point so we can keep going...
- $g = \bigcup_{i<\lambda} g_i : B \to E$.

Applications: Baer-like criterion

The Baer-like criterion for relative injective modules ...

1. is a weakening of Baer's Criterion for injective modules.
2. is new for *RD*-injective modules;
3. was obtained using algebraic methods for pure injective modules in [ŠaTr20]
4. was obtained using algebraic methods for flat cotorsion modules in [ŠaTr20]
5. is new for torsion pure injective modules.

Applications: Enough relative injective modules

Enough injective modules (1940s): Every module can be embedded into an injective module.

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Enough injective modules (1940s): Every module can be embedded into an injective module.

Enough relative injective modules

If $A \in K$, then there is $B \in K$ such that $A \leq_{\mathcal{M}} B$ and B is $K_{\mathcal{M}}$-injective.

Proof.

- Let B be a $(\lambda, (\mathrm{card}(R) + \aleph_0)^+)$-limit model such that $A \leq_{\mathcal{M}} B$. It exists by stabilty.
- B is $K_{\mathcal{M}}$-injective by Baer-like criterion.

Outline

Summary and future work

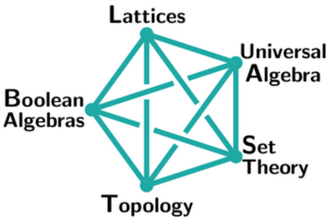
Summary

- There are many natural abstract elementary classes of modules.
- Superstability is a natural algebraic property.
- Abstract elementary classes can be used to answer algebraic questions.

Future work

- Are all abstract elementary classes of modules with pure embeddings stable?
- Can the results be extended to other varieties?
- Use abstract elementary classes of modules to answer algebraic questions.

Invitation: BLAST 2026



- Baylor University at Waco Texas.
- Dates: Summer 2026 (May? August?)

More information before Christmas...

Thank you!

- Marcos Mazari-Armida and Jiří Rosický , *Relative injective modules, superstability and noetherian categories,*, to appear in Journal of Mathematical Logic, 32 pages.
- Marcos Mazari-Armida, *Superstability, noetherian rings and pure-semisimple rings*, Annals of Pure and Applied Logic **172** (2021), no. 3, 102917 (24 pages).
- Marcos Mazari-Armida and Jiří Rosický , *On the abstract elementary class of acts with embeddings*, Preprint.