# Prime Maltsev conditions and compatible digraphs

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## Example

- $\mathcal{SET}$ is the variety of all sets with no operations,
- $\mathcal{SG}$ is the variety of all semigroups $(S; \cdot)$,
- $\mathcal{SLAT}$ is the variety of all semilattices $(S; \wedge)$,
- $\mathcal{BA}$ is the variety of all Boolean algebras $(B; \wedge, \vee, ', 0, 1)$,
- $\mathcal{BR}$ is the variety of all Boolean rings $(R; +, \cdot, 0, 1)$.

The varieties $\mathcal{BA}$ and $\mathcal{BR}$ are not the same, but they are term equivalent. Every Boolean algebra can be turned into a boolean ring and vice versa:

$$x + y = (x \wedge y') \vee (x' \wedge y), \quad x \cdot y = x \wedge y, \quad 1 = 1, \quad 0 = 0.$$

$\mathcal{SET}$ and $\mathcal{SG}$ are not term equivalent, but every set can be turned into a semigroup by $x \cdot y = x$, and vice versa.

Semigroups cannot be turned into semilattices, because semilattices satisfy the identity $x \wedge y = y \wedge x$ and such operation cannot be defined as a semigroup term.

## Definition (W.D. Neumann; 1974)

Let $\Gamma$ be a set of identities over a signature. We say that $\Gamma$ **interprets in a variety** $\mathcal{K}$ if by replacing the operation symbols in $\Gamma$ by some term expressions of $\mathcal{K}$, the so obtained set of identities holds in $\mathcal{K}$.

## Definition

A **variety** $\mathcal{K}_1$ **interprets in a variety** $\mathcal{K}_2$, denoted as $\mathcal{K}_1 \preceq \mathcal{K}_2$, if there is a set of identities $\Gamma$ that defines $\mathcal{K}_1$ and interprets in $\mathcal{K}_2$.

- The varieties $\mathcal{BA}$ and $\mathcal{BR}$ are equi-interpretable.
- The varieties $\mathcal{SET}$ and $\mathcal{SG}$ are equi-interpretable.
- The variety of groups interprets in the variety of Abelian groups.
- The variety $\mathcal{SET}$ interprets in any other variety.
- Every variety interprets in the variety of trivial algebras ($x \approx y$).
- Constants $c$ are modelled by unary operations satisfying $c(x) \approx c(y)$.
- The interpretability relation $\preceq$ is a quasi-order on the class of varieties.

# Lattice of interpretability types

## Theorem

*The class of varieties modulo equi-interpretability forms a bounded lattice,* **the lattice of interpretability types**, *with* $\overline{\mathcal{V}} \vee \overline{\mathcal{W}} = \overline{\mathcal{V} \amalg \mathcal{W}}$ *and* $\overline{\mathcal{V}} \wedge \overline{\mathcal{W}} = \overline{\mathcal{V} \otimes \mathcal{W}}$.

## Definition

The **coproduct** of the varieties $\mathcal{V} = \operatorname{Mod}\Sigma$ and $\mathcal{W} = \operatorname{Mod}\Delta$ in disjoint signatures is the variety $\mathcal{V} \amalg \mathcal{W} = \operatorname{Mod}(\Sigma \cup \Delta)$.

## Definition

The **varietal product** of $\mathcal{V}$ and $\mathcal{W}$ is the variety $\mathcal{V} \otimes \mathcal{W}$ of algebras $\mathbf{A} \otimes \mathbf{B}$ for $\mathbf{A} \in \mathcal{V}$ and $\mathbf{B} \in \mathcal{W}$ whose

- universe is $A \times B$,
- basic operations are $s \otimes t$ acting coordinate-wise for each pair of $n$-ary terms of $\mathcal{V}$ and $\mathcal{W}$.

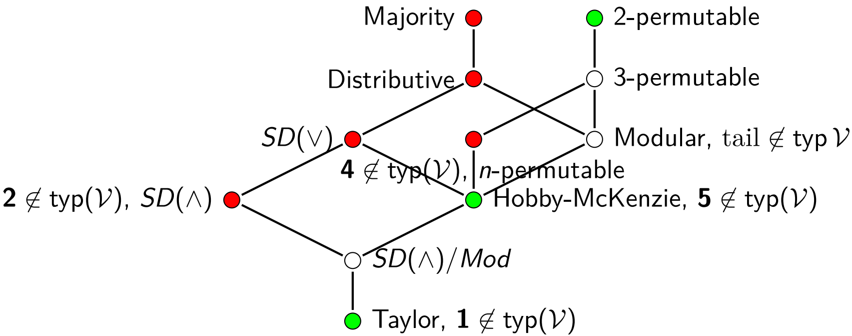
## Theorem (O. Garcia, W. Taylor; 1984)

*In the lattice of interpretability types of varieties*

- *minimal element: sets, maximal element: trivial algebras*
- *the class of idempotent varieties form a sublattice*
- *the class of finitely presented varieties forms a sublattice*
- *the class of linear varieties forms a join sub-semilattice*
- *join prime elements: commutative groupoids, trivial algebras*

- O. Garcia and W. Taylor **conjectured** in 1984 that congruence permutability is a join prime element.
- L. Sequeira proved it for **linear** varieties.
- S. Tschantz announced a proof of the conjecture in 1996.
- K. Kearnes and S. Tschantz proved it for **idempotent** varieties.
- M. Valeriote and R. Willard: $n$-permutability for some $n$ is a prime filter for **idempotent** varieties.
- J. Opršal: for any $n \geq 2$, $n$-permutability is prime for **linear** varieties.

# Maltsev filters of varieties




- prime Maltsev filters:
  - congruence permutable, $m(x, y, y) \approx m(y, y, x) \approx x$
  - Hobby-McKenzie term (join semi-distributive over modular)
  - Taylor term, non-trivial idempotent Maltsev condition
- non-prime Maltsev filters:
  - congruence $n$-permutable for some $n$, Hagemann-Mitschke terms
  - congruence distributive = join semi-distributive and modular
  - congruence join semi-distributive (K. Kearnes and E. W. Kiss)

# Majority is not prime

**Theorem**

*Congruence meet semi-distributivity, congruence join semi-distributivity, congruence distributivity and having a majority term are not prime in the lattice of interpretability types of varieties.*

- Let $\mathcal{V}$ be the variety defined by the minority identities
$$m(x, y, y) \approx m(y, x, y) \approx m(y, y, x) \approx x.$$
- Let $\mathcal{W}$ be the variety defined by identities
$$s(x, x) \approx x, \qquad s(x, y) \approx s(y, x).$$
- We have $\mathbf{A} = (\mathbb{Z}_2; x + y + z) \in \mathcal{V}$ and $\mathbf{B} = (\mathbb{Z}_3; 2x + 2y) \in \mathcal{W}$. Con $\mathbf{A}^2 \cong \mathbf{M}_3$ and Con $\mathbf{B}^2 \cong \mathbf{M}_4$, so $\mathcal{V}$ and $\mathcal{W}$ are not congruence meet semi-distributive.
- However, their join has a majority term:
$$m(s(x, y), s(y, z), s(z, x)).$$

# Taylor is prime

**Theorem (W. Taylor, 1977)**

*For any variety $\mathcal{V}$ the following are equivalent*

- $\mathcal{V}_{\text{id}} \not\leq \mathcal{SET}$,
- *satisfies a non-trivial idempotent Maltsev condition,*
- *has a Taylor-term:* $t(x, \dots, x) \approx x$ *and* $t(\dots, x, \dots) \approx t(\dots, y, \dots)$.

**Theorem**

*The filter of Taylor varieties is prime in the lattice of interpretability types.*

Approach: Given two non-Taylor varieties $\mathcal{V}$ and $\mathcal{W}$, find a compatible digraph $\mathbb{G}$ in both $\mathcal{V}$ and $\mathcal{W}$ that does not admit a Taylor polymorphism.

**Lemma**

*Let* $\mathbb{C} = (\{0, 1, 2\}; \to)$ *be the reflexive directed 3-cycle. A variety is non-Taylor iff it has a compatible reflexive digraph that has* $\mathbb{C}$ *as a retract.*

## Theorem

*For any variety $\mathcal{V}$ the following are equivalent:*

- $\mathcal{V}$ *is non-Taylor,*
- *there are sets $K_t$ $(t \in T)$, not all empty, such that $\dot{\bigcup}_{t\in T}\mathbb{C}^{K_t}$ is a compatible digraph in $\mathcal{V}$,*
- *for any sufficiently large infinite cardinals $\kappa$ and $\tau$ the digraph $\dot{\bigcup}_{\mu\leq\kappa}\tau\mathbb{C}^{\mu}$ is a compatible digraph in $\mathcal{V}$.*

## Proposition

- $\mathcal{SET} \lneq \mathrm{Pol}(\mathbb{C})$ *because of the Maltsev condition $u(x) \approx u(y)$.*
- $\mathrm{Pol}(\mathbb{C}) \lneq \mathrm{Pol}(\mathbb{C}^2)$ *because of the Maltsev condition*

  $$f(f(x,y), f(y,z)) \approx y$$

  *satisfied by the polymorphism $f(\overline{x_1x_2}, \overline{y_1y_2}) = \overline{x_2y_1}$ of $\mathbb{C}^2$.*
- $\mathrm{Pol}(\mathbb{C}+1) \not\leq \mathrm{Pol}(\mathbb{C}^K)$ *because of the Maltsev condition*

  $$e(x) \approx e(y), \quad t(e(x), y) \approx t(y, e(x)) \approx y.$$

# Hobby-McKenzie is prime

## Theorem (D. Hobby and R. McKenzie; 1988)

*For any variety $\mathcal{V}$ the following are equivalent:*

- *$\mathcal{V}$ has Hobby-McKenzie terms,*
- *$\mathcal{V}_{\mathrm{id}} \not\preceq \mathcal{SLAT}$.*

## Theorem

*Let $\mathbb{S}$ be a connected reflexive relational structure and $\mathcal{S}$ be the variety generated by $(S; \mathrm{Pol}(\mathbb{S}))$. If $\mathcal{V}_{\mathrm{id}} \preceq \mathcal{S}$, then $\mathcal{V}$ has a compatible relational structure $\mathbb{F}$ with $\mathbb{S}$ as a retract.*

For the variety $\mathcal{SLAT}$ we have considered the following two structures:

$$\mathbb{D} = (\{0,1,2\}; \{00,11,22,01,10,12,20\}),$$
$$\mathbb{S} = (\{0,1\}; \{000,010,100,111\}).$$

Similar theorems hold as for Taylor varieties.

# Congruence and graph conditions

**Congruence condition**: any property of varieties that can be expressed by the congruence relations of the algebras in the variety.

**Graph condition**: any property of varieties that can be expressed by the set of compatible directed graphs of the algebras in the variety.

## Proposition

*A variety $\mathcal{V}$ is congruence 2-permutable iff every reflexive compatible digraph in $\mathcal{V}$ is symmetric (and transitive).*

## Definition

The **extreme congruence** of a digraph $\mathbb{G} = (G; \to)$ is $(\to \cap \leftarrow)^*$, the **strong congruence** is $\to^* \cap \leftarrow^*$, the **weak congruence** is $(\to \cup \leftarrow)^*$.

## Proposition

*A variety $\mathcal{V}$ is congruence $n$-permutable for some $n$ iff the strong and weak congruences are the same in every reflexive compatible digraph in $\mathcal{V}$.*

# Graph conditions for Taylor varieties

**Theorem**

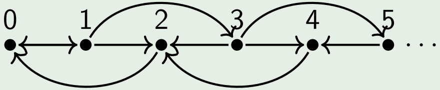
*A variety is Taylor iff all its reflexive antisymmetric digraphs are cycle free.*

**Theorem**

*A variety $\mathcal{V}$ is Taylor iff the $*$-extreme and strong congruences are the same in every compatible reflexive digraph in $\mathcal{V}$, where the* **$*$-extreme congruence** *is the smallest equivalence that makes the factor antisymmetric.*

**Example**

The following digraph has a compatible semilattice with linear order, so need to factorize by the extreme congruence arbitrary many times.



## Definition

A subset $X$ of an algebra $\mathbf{A}$ is **strongly rectangular** if for every $n$-ary term $t$ and all elements $x_i, y_i \in X$ and $z_i \in \{x_i, y_i\}$ we have

$$t(x_1, \ldots, x_n) = t(y_1, \ldots, y_n) \Rightarrow t(x_1, \ldots, x_n) = t(z_1, \ldots, z_n).$$

## Definition

A subset $X$ of an algebra $\mathbf{A}$ is **strongly Abelian** if for every $n$-ary term $t$ and all elements $x_i, y_i, z_i \in X$ we have

$$t(x_1, \ldots, x_n) = t(y_1, \ldots, y_n) \Rightarrow t(x_1, z_2, \ldots, z_n) = t(y_1, z_2, \ldots, z_n).$$

## Definition

A subset $X$ of an algebra $\mathbf{A}$ is **diagonal** if for every $n$-ary term $t$, and all elements $x_{1,1}, x_{1,2}, \ldots, x_{n,n} \in X$ we have

$$t(t(x_{1,1}, \ldots, x_{1,n}), \ldots, t(x_{n,1}, \ldots, x_{n,n})) = t(x_{1,1}, x_{2,2}, \ldots, x_{n,n}).$$

## Theorem

*For idempotent algebras the above three concepts are all equivalent.*

# Graph conditions for other varieties

**Theorem**

*A variety $\mathcal{V}$ has Hobby-McKenzie terms iff the strong and extreme congruences are the same in every reflexive compatible digraph in $\mathcal{V}$.*

**Theorem (A. Kazda; 2011)**

*Any finite digraph that admits a Maltsev operation also admits a majority operation.*

**Corollary**

*The variety $\mathcal{M}$ admitting a majority operation cannot be described by a graph condition referencing only finite graphs.*

**Theorem (M. M. and L. Zádori; 2012)**

*Any finite reflexive digraph that admits Gumm operations also admits a near-unanimity operation.*

# Bibliography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