Algebras of conditionals

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Introduction

A **conditional statement** is a hypothetical proposition of the form "If [antecedent] is the case, then [consequent] is the case", where the antecedent is assumed to be true. Such a notion can be formalized by expanding the language of classical logic by a binary operator y/x that reads as *"y given x"*.

Conditional expressions are central in representing knowledge and reasoning abilities of intelligent agents. Conditional reasoning indeed features in a wide range of areas including non-monotonic reasoning, causal inference, learning, and more generally reasoning under uncertainty.

Introduction

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In this contribution, we propose a totally different approach to formalizing conditional statements based instead on a purely algebraic intuition. The idea on which we ground our proposal is that, in the algebraic setting of Boolean algebras, there is a natural way of formalizing such a statement, starting from the notion of quotient.

Algebraic intuition

Given a Boolean algebra $\mathbf{B} = \langle B, \wedge, \vee, \neg, 0, 1\rangle$ and an element $x \in B$, one can consider the congruence that collapses x and the truth constant 1, and then denote by $\mathbf{B}/x$ the quotient of $\mathbf{B}$ by this congruence. Intuitively, this corresponds to assuming that x is true.

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Now we wish to define a conditional operator on $\mathbf{B}$ such that y/x represents the element y as seen in the quotient $\mathbf{B}/x$.
The particular structural properties of Boolean algebras allow us to do so in a natural way.

Algebraic intuition

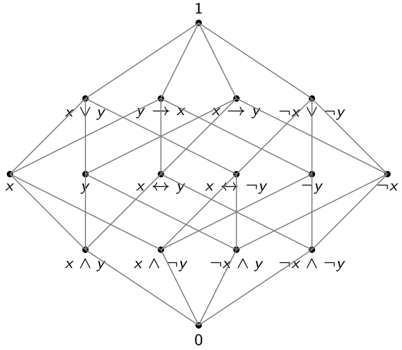
Take $x \in \mathbf{B}$ such that $x \neq 0$, then the quotient $\mathbf{B}/x$ is actually a retract of $\mathbf{B}$, i.e. it is isomorphic to a subalgebra of $\mathbf{B}$ and, if we call π_x the natural epimorphism $\pi_x : \mathbf{B} \to \mathbf{B}/x$, there is an injective homomorphism $\iota_x : \mathbf{B}/x \to \mathbf{B}$ such that

$$\pi_x \circ \iota_x = \mathrm{id}_{\mathbf{B}/x}. \qquad \textit{(Identity condition)}$$

Notice that the map ι_x is not uniquely determined.

The idea is then to consider $y/x = \iota_x \circ \pi_x(y)$.

Algebraic intuition



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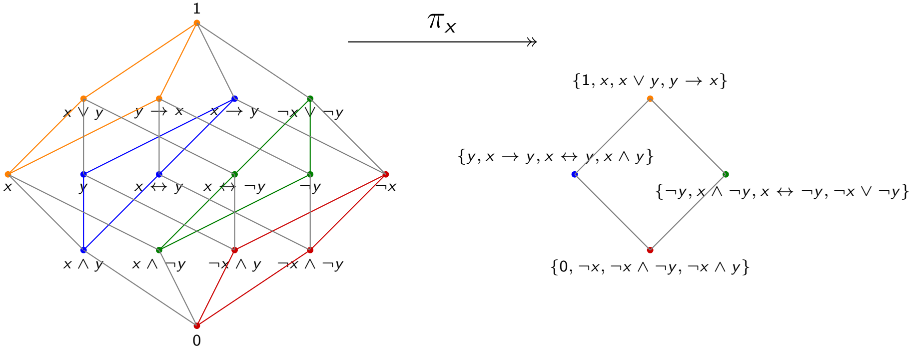

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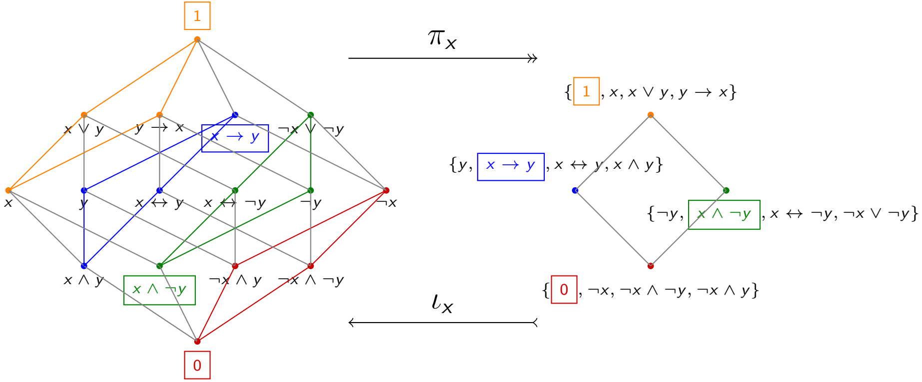

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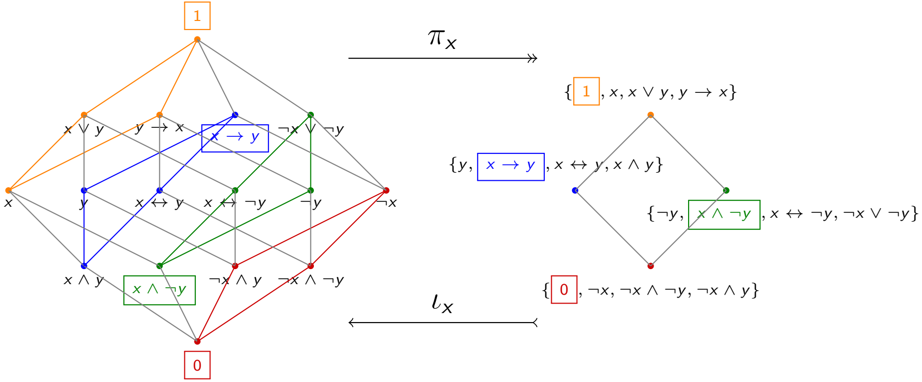

$$y/x = \iota_x \circ \pi_x(y) = x \to y$$

Algebraic intuition

Now, in order to be able to define an operator $/$ over the whole algebra $\mathbf{B}$, one needs to consider all the different principal quotients. Then, note that if $0 \neq x \leq z$, we get that $(\mathbf{B}/z)/\pi_z(x) \cong \mathbf{B}/x$, in other words, $\mathbf{B}/x$ is a quotient of $\mathbf{B}/z$. The situation is the following:

$$
\begin{tikzcd}
\mathbf{B} \arrow[r, two heads, "\pi_x"] \arrow[d, two heads, "\pi_z"'] & \mathbf{B}/x \\
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\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbf{B} & \mathbf{B}/x \arrow[l, tail, "\iota_x"'] \arrow[dl, tail, "\iota_{\pi_z(x)}"] \\
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$$

It is then natural to ask that the choices for ι_x and ι_z be compatible, in the sense that there is a way of choosing the embedding $\iota_{\pi_z(x)}$ so that the second diagram commutes, and so

$$\iota_x = \iota_z \circ \iota_{\pi_z(x)}, \qquad \textit{(Compatibility condition)}$$

which yields $y/x = (y/x)/z$ (whenever $x \leq z$).

Algebraic intuition

It is left to discuss the case where $x = 0$. Note that, in this case, the associated quotient is the trivial algebra $\mathbf{T}$, where all elements are identified. This is not isomorphic to a subalgebra of any nontrivial Boolean algebra $\mathbf{B}$ since there cannot be an embedding from $\mathbf{T}$ to $\mathbf{B}$, given that the constants 0 and 1 are identified in $\mathbf{T}$. Thus, all elements of π_0 are collapsed to a singleton and should be mapped together to a constant.

Since intuitively we are considering the quotient by an element x to mean that "x is true", the *ex falso quodlibet* suggests that we should map all elements to 1, i.e. $y/0 = 1$.

Algebraic properties

The so-defined operation $/$ has some interesting properties on these standard models. For example, it is not difficult to see that it holds:

1. $1/x = 1$
2. $x/1 = x$
3. $x \wedge y \leq y/x \leq x \to y$
4. $(y \wedge z)/x = (y/x) \wedge (z/x)$

Moreover, if $x \neq 0$

5. $(\neg y)/x = \neg(y/x)$
6. $0/x = 0$.

Dual standard models

Summing up, in order to obtain a standard model, we need a Boolean algebra $\mathbf{B}$ that serves as the Boolean domain and a class of embeddings that satisfy the identity condition and the compatibility condition.

We now use Stone duality to translate the above conditions to the dual setting; in other words, we generate the standard models as algebras of sets. From now on, let us assume that $\mathbf{B}$ is finite.
By the finite version of Stone duality, we now see the algebra $\mathbf{B}$ as an algebra of sets, say that $\mathbf{B} = \mathcal{S}(X)$, where $\mathcal{S}(X) = \langle \mathcal{P}(X), \cap, \cup, ^{C}, \emptyset, X \rangle$ is a Boolean algebra isomorphic to $\mathbf{B}$.

Dual standard models

Now, the previous reasoning translates to the following:

- given $Y \subseteq X$, the natural epimorphism $\pi_Y : \mathcal{S}(X) \to \mathcal{S}(Y)$ dualizes to the identity map $\mathrm{id}_Y : Y \to X$;
- the embedding $\iota_Y : \mathcal{S}(Y) \to \mathcal{S}(X)$ dualizes to a surjective map $f_Y : X \to Y$;
- the identity condition now becomes $f_Y \circ \mathrm{id}_Y = \mathrm{id}_Y$, i.e. we are asking that f_Y restricted to Y is the identity;
- the compatibility condition translates on the dual in, given $Y \subseteq Z \subseteq X$, then $f_Y = f_Y \circ f_Z$.

Dual standard models

Let us be more precise:

Definition

Given a set X, we say that a class of surjective functions $\mathcal{F} = \{f_Y : X \to Y : \emptyset \neq Y \subseteq X\}$ is **compatible with X** if, for every $Y \subseteq Z \subseteq X$:

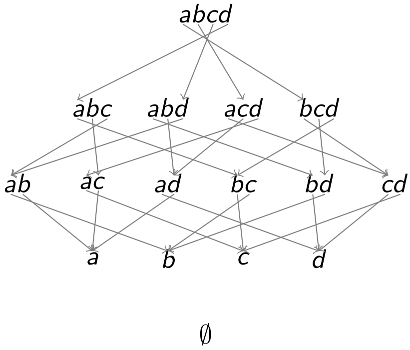
- f_Y restricted to Y is the identity on Y;
- $f_Y = f_Y \circ f_Z$.

A **standard model** is a Boolean algebra of sets $\mathcal{S}(X)$ for some set X, with $/$ defined starting from a class of functions $\mathcal{F}$ compatible with X in the following way:

$$Y/Z = (f_Z)^{-1}(Y \cap Z)$$

for any $Y, Z \subseteq X$ and $Z \neq \emptyset$. Moreover $Y/\emptyset = X$.

Dual standard models



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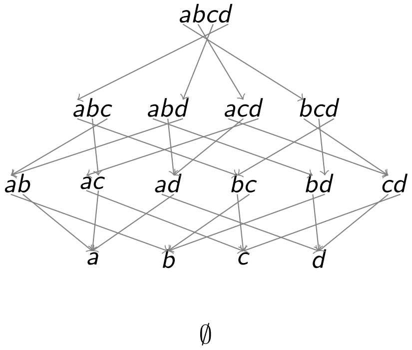

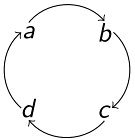

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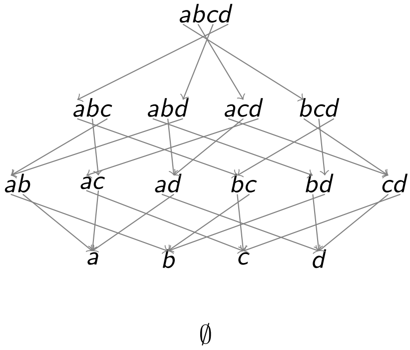

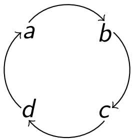

$$ac/ab = {f_{ab}}^{-1}(ab \cap ac) = {f_{ab}}^{-1}(a) = acd$$

Stalnaker-Lewis models

The best-known semantics for conditional operators is probably the one given by Stalnaker and Lewis.

Definition

Given a finite set of possible worlds Ω, a **sphere frame** on Ω is a system $\Sigma = (\Omega, \mathcal{S})$, where $\mathcal{S}$ is a function from Ω to $\mathcal{P}(\mathcal{P}(\Omega))$, that assigns to each $\alpha \in \Omega$ a set $\mathcal{S}_\alpha$ of nested subsets of Ω.
Moreover, a sphere frame is

- **centered** if for all $\alpha \in \Omega$, $\bigcap \mathcal{S}_\alpha = \{\alpha\}$;
- **uniform** if for all $\alpha \in \Omega$, $\bigcup \mathcal{S}_\alpha = \Omega$;
- **Stalnaker** if at every sphere we add exactly one element;
- **strict Stalnaker** if it is centered, uniform and Stalnaker.

Stalnaker-Lewis models

In our case, we can assume the set of possible worlds to be the finite set of atoms $X = \{a, b, c, \ldots\}$ of our finite Boolean algebra $\mathcal{S}(X)$.

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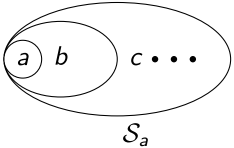

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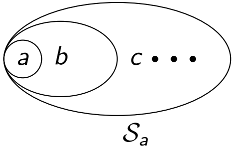

Every strict Stalnaker frame over a finite set X can also be represented with a family of total and complete orders on X, $\{\leq_a\}_{a\in X}$, such that $a \leq_a b$ for every $b \in X$.
Given a finite strict Stalnaker frame $\{X, \{\leq_a\}_{a\in X}\}$, Stalnaker and Lewis define a conditional operator on X as

$$Y/Z = \{a \in X \mid \min_a(Z) \in Y\}.$$

Stalnaker-Lewis models

We can show that our standard models are equivalent to the class of strict Stalnaker frames satisfying a further condition.
Indeed, given a finite set X and a class of functions $\mathcal{F}$ compatible with X, we define for every $a \in X$ the order

$$a \leq_a \underbrace{f_{X-\{a\}}(a)}_{=\alpha} \leq_a f_{X-\{a\}-\{\alpha\}}(a) \leq_a \dots,$$

and we get a strict Stalnaker frame satisfying for every $a \in X$ and for every $Y \subseteq Z \subseteq X$, $\min_a(Y) = \min_{\min_a(Z)}(Y)$.
Conversely, given a strict Stalnaker frame $(X, \{\leq_a\}_{a \in X})$ such that for every $a \in X$ and for every $Y \subseteq Z \subseteq X$ it holds that $\min_a(Y) = \min_{\min_a(Z)}(Y)$, if we define

$$f_Y(a) = \min_a(Y),$$

then the class $\mathcal{F} = \{f_Y : \emptyset \neq Y \subseteq X\}$ is compatible with X.
Moreover, the two definitions of conditional are exactly the same on X.

The variety QA

Following this intuition, we proved that the variety generated by our standard models (finite and infinite ones) is indeed a subvariety of the variety of strict Stalnaker algebras that has been axiomatized by Lewis.

Theorem

Let QA be the variety generated by all standard models; then QA is axiomatized modulo strict Stalnaker algebras by the axiom

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Now that we have an axiomatization for this variety we can study some of its properties. For example we proved that QA is a **discriminator variety**.

The variety QA

An algebra $\mathbf{A}$ with a Boolean reduct is a **discriminator algebra** if there exists a unary term c such that in $\mathbf{A}$ it holds

$$c(x) = 0 \text{ if } x = 0 \qquad \text{and} \qquad c(x) = 1 \text{ if } x \neq 0.$$

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Corollary

In QA *the concepts of an algebra being simple, subdirectly irreducible or directly indecomposable are equivalent. Moreover all these classes coincide with the class of all standard models.*

Conclusions and future (and current) works

Following a purely algebraic intuition, we described the class of our standard models both in the algebraic and in the dual setting. Moreover, we found that the variety generated by all standard models QA is a subvariety of strict Stalnaker algebras and in particular is axiomatized modulo the latter by the axiom

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- Study more properties of this variety (e.g. finite model property, locally finiteness...).
- Characterize free algebras.
- Define and study a notion of probability over these structures.

Thanks for your attention!