The Monadic Grzegorczyk Logic

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There have been many attempts to remedy this.

One such is to consider Kripke bundles, which generalize predicate Kripke frames [Shehtman, Skvortsov, 1990].

Definition

A Kripke bundle is a triple $((X,R),\pi,(X_0,R_0))$, where (X,R), (X_0,R_0) are Kripke frames, and $\pi\colon (X,R)\to(X_0,R_0)$ is an onto p-morphism.

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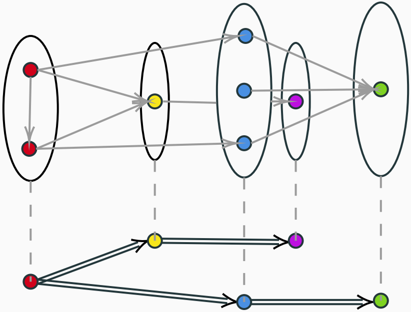

$$\begin{array}{c}(X, R)\\ \Big\downarrow{\scriptstyle \pi}\\ (X_0, R_0)\end{array}$$

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1. One approach would be to develop a more sophisticated semantics that matches pm-logics in terms of complexity (e.g., hyperdoctrines and metaframes).
2. The other approach is to reduce the expressivity of the logics to match the semantics we have available.

Monadic modal logic

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We assume that $\exists$ is an **S5**-modality, meant to stand for existential quantification over a fixed variable.

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Definition

A propositional logic in the signature $\{\Diamond, \exists\}$ is a monadic modal logic (mm-logic) if $\exists$ is an **S5**-modality and the two modalities are connected by the axiom $\exists\Diamond p \to \Diamond\exists p$.

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$$(\forall x, y, z \in X)(x \; E \; y \text{ and } x \; R \; z \Longrightarrow \exists u \in X : y \; R \; u \text{ and } u \; E \; z).$$

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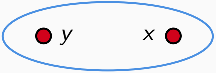

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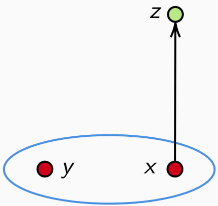

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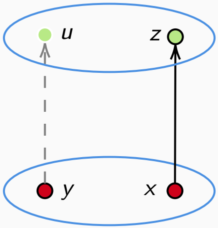

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- The formula $\exists x \Diamond p(x) \to \Diamond \exists x p(x)$ is sometimes referred to as the converse Barcan formula.

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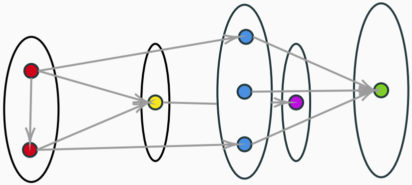

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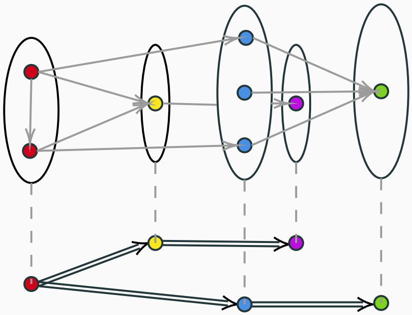

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This is an object level correspondence between **MK**-frames and Kripke bundles, which extends to an equivalence of the corresponding categories.

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- The above theorem yields many other examples of mm-logics that are monadic fragments of the corresponding pm-logics.
- It is less clear whether an analogous result is true for **MGrz** and **QGrz** since the completeness of **MGrz** is not so obvious.

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Corollary

1. **MGrz** is complete.
2. **MGrz** is decidable.
3. **MGrz** is the monadic fragment of **QGrz**.

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The construction we propose is a refinement of these and is based on the idea of a "strongly maximal" point.

Strongly maximal points

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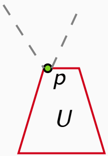

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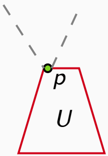

But maximal points do not take E into account.

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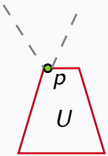
Given $U \subseteq X$, we say $p \in U$ is maximal in U if $(R[p] - \{p\}) \cap U = \varnothing$.

In other words, a proper R move from p leaves one outside of the set U.



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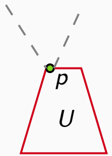

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This means that a proper R move from p and moving "sideways" through E leaves one out of U.

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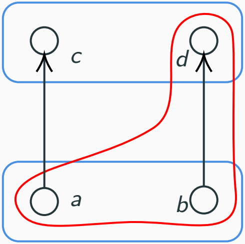

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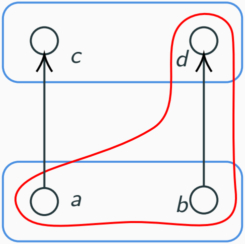

The red curve encloses a subset of the frame. The point a is maximal in the indicated subset, but not strongly maximal. On the other hand, d is a strongly maximal point.

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This construction terminates with a finite counter-model for φ.

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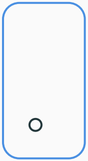

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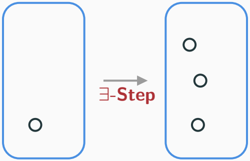

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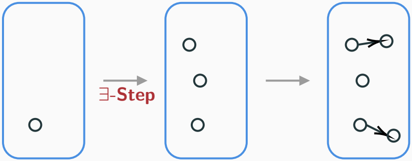

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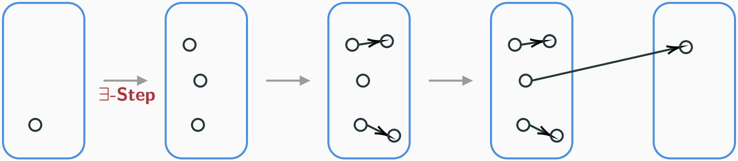

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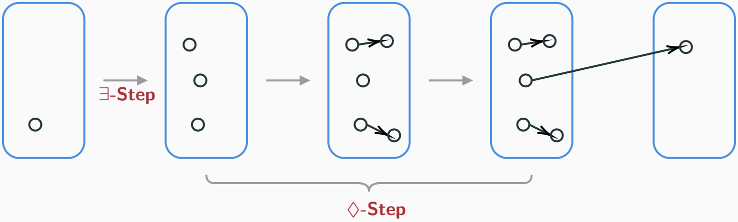

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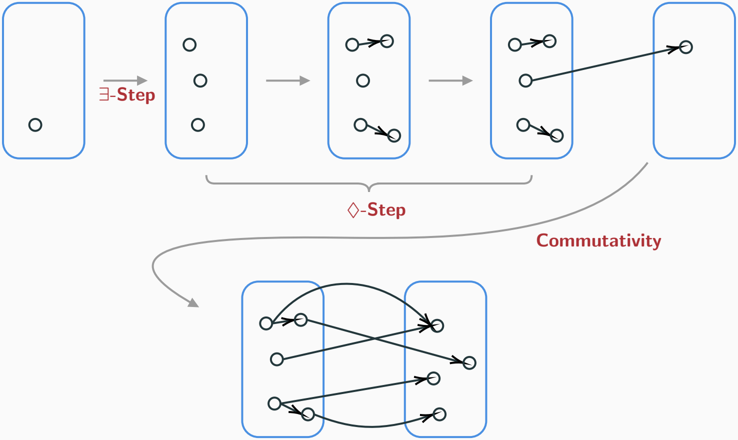

Consequences

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- The standard filtration of Goranko and Passy does not apply in the case of $\mathbf{Grz}_u$, so the above cannot be obtained using their results.

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To remedy this, he proposed the system $\mathbf{Q^{+}Grz}$ by considering the Gödel translation of Casari's schema:

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From this perspective, $\mathbf{M^{+}Grz}$ embeds into **MGL**, and thus has a provability interpretation.

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1. $\mathbf{M^+Grz}$ has the fmp.
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This yields a unified approach to the results of [Japaridze, 1990] and [G. Bezhanishvili, Brantley, Ilin, 2023].

Future work

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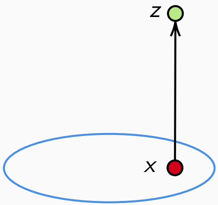

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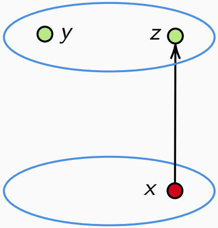

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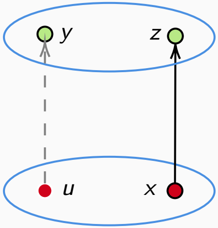

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Our construction would thus require a significant revision to accommodate the Barcan formula.

Thank you!