The category of linear modular lattices: exploring the lattice-theoretic counterparts of modules

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Modules and Lattices

Given an R-module M, we can always define the lattice $L(M)$ by

$$L(M) = \{A \mid A \text{ is an } R\text{-submodule of } M\}.$$

Here, the join operation is given by $A + B$, while the meet operation is given by $A \cap B$. Furtermore, this lattice satisfies the following properties:

- $L(M)$ is complete. That is, every subset $S \subseteq L(M)$ has a join and a meet.
- The greatest and smallest elements are M and 0, represented by 1_M and 0_M, respectively.
- $L(M)$ is modular. That is, for all elements $a, b, c \in L(M)$, the following implication holds:

$$a \leq c \Longrightarrow (a \vee b) \wedge c \leq a \vee (b \wedge c).$$

Module and lattice morhpisms

An *R-module morphism* is a function $f : M_1 \longrightarrow M_2$ that preserves the module structure. That is,

- $f(x+y) = f(x) + f(y)$, for $x, y \in M_1$;
- $f(rx) = rf(x)$, for $x \in M_1$ and $r \in R$.

A *lattice morphism* is a function $g : L_1 \longrightarrow L_2$ that preserves the lattice operations of meet and join. That is,

- $g(x \wedge y) = g(x) \wedge g(y)$, for $x, y \in L_1$;
- $g(x \vee y) = g(x) \vee g(y)$, for $x, y \in L_1$.

Remark

For an R-module morphism $f : M_1 \longrightarrow M_2$, one always has $f(A+B) = f(A) + f(B)$. Thus, we always obtain an induced $\vee$-morphism $g : L(M_1) \longrightarrow L(M_2)$ in their corresponding lattices.

Linear Lattice Morphisms

Definition

We call a function $f : L \longrightarrow L'$ a linear lattice morphism if there exits $k \in L$, called the kernel of f, and $a' \in L'$ such that the following holds:

a) $f(x) = f(x \vee k), \forall x \in L.$

b) f *induces a lattice isomorphism* $\bar{f} : 1_L/k \xrightarrow{\approx} a'/0_{L'}$*, such that*

$$\bar{f}(x) = f(x), \forall x \in 1_L/k.$$

Not every lattice morphism, in the usual sense, is a linear lattice morphism:

Example

The function $[1,2] \overset{\iota}{\hookrightarrow} [0,3]$ is a lattice morphism in the usual sense; however, it is not a linear lattice morphism.

Proposition

For each R-module morphism $f : M_1 \longrightarrow M_2$, one has an induced linear lattice morphism $\varphi_f : L(M_1) \longrightarrow L(M_2)$ given by

$$\varphi_f(A) = f(A), \text{ for any } A \in L(M_1).$$

Here, the kernel of φ_f is $\ker(f)$.

The category of linear modular lattices $\mathcal{L}_{\mathcal{M}}$

We denote by $\mathcal{L}_{\mathcal{M}}$ the category whose objects are complete modular lattices and whose morphisms are linear morphisms between complete modular lattices.

Some facts about $\mathcal{L}_{\mathcal{M}}$ and its linear lattice morphisms:

- The monomorphisms in the category $\mathcal{L}_{\mathcal{M}}$ are exactly the injective linear morphisms;
- The epimorphisms in the category $\mathcal{L}_{\mathcal{M}}$ are exactly the surjective linear morphisms;
- The subobjects of $L \in \mathcal{L}_{\mathcal{M}}$ can be taken as the intervals $a/0_L$ for any $a \in L$.

Hereditary and Natural classes of R-modules

Definition

Given an associative ring R, we say that a class $\mathcal{C}$ of R-modules is hereditary if $\mathcal{C}$ is closed under submodules. That is,

$$\text{For any } M \in \mathcal{C} \text{ and } N \leq M, \text{ one has that } N \in \mathcal{C}.$$

Definition

A class $\mathcal{C}$ of R-modules is called natural if it is hereditary, closed under injective envelopes, and closed under direct sums. That is,

(i) *If $N \leq M \in \mathcal{C}$, then $N \in \mathcal{C}$;*

(ii) *If $M \in \mathcal{C}$, then $E(M)$[1] $\in \mathcal{C}$;*

(iii) *If $M_i \in \mathcal{C}$, for $i \in I$, then $\bigoplus_{i\in I} M_i \in \mathcal{C}$.*

[1] The injective hull of a module is both the smallest injective module containing it and the largest essential extension of it.

Main results

Definition

For a lattice L, we say that b is a pseudocomplement of a if $a \wedge b = 0_L$ and b is maximal with this property.

Definition

The skeleton of a pseudocomplemented lattice L is the class of all pseudocomplements of L ([3, Definition 11]).

Theorem

The collection of all hereditary classes of R-modules forms a complete pseudocomplemented big lattice, denoted by R-her.

Theorem

The collection of natural classes in R-Mod, denoted by R-nat, is the skeleton of R-her. Further, R-nat is a complete Boolean lattice.

The lattice-theoretic counterpart

From now on, we will consider the subcategory $\mathcal{L}_{\mathcal{M}c}$ of complete upper semi-continuous modular lattices and linear morphisms.

Definition

We say that a lattice L is upper semi-continuous if for every directed subset D of L and any $a \in L$, one has that

$$\left(\bigvee_{d \in D} d\right) \wedge a = \bigvee_{d \in D} \left(d \wedge a\right).$$

(Nonetheless, most of the results also hold for lattices in the broader category $\mathcal{L}_{\mathcal{M}}$)

Hereditary classes of linear lattices

Definition

[2, Albu] We say that a class $\mathcal{C}$ in $\mathcal{L}_{\mathcal{M}_c}$ is abstract if $\mathcal{C}$ is closed under isomorphisms. That is, if $L \in \mathcal{C}$ and $N \cong L$, then $N \in \mathcal{C}$.

Definition

[2, Albu] We say that a class $\mathcal{C}$ in $\mathcal{L}_{\mathcal{M}_c}$ is hereditary if $\mathcal{C}$ is abstract, and if for every $L \in \mathcal{L}_{\mathcal{M}_c}$ and $a \leq b \leq c$ in L, such that $(c/a) \in \mathcal{C}$, we get that $(b/a) \in \mathcal{C}$.

Some examples of hereditary classes of lattices are:

- The class of all atomic modular lattices;
- The class of all modular lattices with finite Goldie dimension;
- The class of all modular noetherian lattices;
- The class of all modular artinian lattices.

Let us denote by $\mathcal{L}_{her}$ the collection of all hereditary classes of lattices in $\mathcal{L}_{\mathcal{M}_c}$.

Remark

⋆ *$\mathcal{L}_{her}$ is ordered by inclusion;*

⋆ *$\mathcal{L}_{her}$ becomes a big lattice when we define the infimum and supremum of any familiy $\{\mathcal{C}_i\}_{i\in I}$ of hereditary classes as*

$$\curlywedge_{i\in I} \mathcal{C}_i = \cap_{i\in I} \mathcal{C}_i \text{ y } \curlyvee_{i\in I} \mathcal{C}_i = \cup_{i\in I} \mathcal{C}_i,$$

respectively.

Pseudocomplements in $\mathcal{L}$-her

Proposition

Let $\mathcal{C}$ be a hereditary class in $\mathcal{L}_{\mathcal{M}_c}$. Then, the class

$$\mathcal{A}_\mathcal{C} = \{L \in \mathcal{L}_{\mathcal{M}_c} \mid \forall\, 0_L \neq a \in L, \exists\, 0 < b \leq a \text{ such that } (b/0_L) \notin \mathcal{C}\}$$

is an hereditary class in $\mathcal{L}_{\mathcal{M}_c}$.

Proposition

Let $\mathcal{C}$ be a hereditary class in $\mathcal{L}_{\mathcal{M}_c}$. Then,

$$\mathcal{C} \cap \mathcal{A}_\mathcal{C} = \{0\}.$$

$\mathcal{L}_{her}$ is a strong pseudocomplemented big lattice

Proposition

Let $\mathcal{C}$ be an hereditary class in $\mathcal{L}_{\mathcal{M}_c}$. If $\mathcal{D} \subseteq \mathcal{L}_{\mathcal{M}_c}$ is a hereditary class such that $\mathcal{C} \cap \mathcal{D} = \{0\}$, then,

$$\mathcal{D} \subseteq \mathcal{A}_{\mathcal{C}}.$$

Theorem

Let $\mathcal{C} \in \mathcal{L}_{her}$. Then,

$$\mathcal{C}^{\perp_{\leq}} = \mathcal{A}_{\mathcal{C}}$$

is a strong pseudocomplement of $\mathcal{C}$ in $\mathcal{L}_{her}$.

Corollary

$\mathcal{L}_{her}$ is a strong pseudocomplemented big lattice.

Closure definitions for linear modular lattices

Recall that in a lattice L, we say that an element a is *essential* if $a \wedge b \neq 0_L$ for any $b \neq 0_L$ in L.

Definition

We say that a lattice L' is an essential extension of L if there exist a linear monomorphism $\varphi : L \longrightarrow L'$ such that $\varphi(1_L)$ is an essential element in L'.

Definition

We say that the class $\mathcal{C}$ in $\mathcal{L}_{\mathcal{M}_c}$ is closed under essential extensions if for every $L \in \mathcal{C}$, any essential extension of L belongs to $\mathcal{C}$.

We say that a non empty subset A in a complete modular lattice L is *independent* if, and only if, for all $a \in A$,

$$a \wedge \bigvee \left(A \backslash \{a\}\right) = 0_L.$$

Definition

We say that a class $\mathcal{C}$ in $\mathcal{L}_{\mathcal{M}_c}$ is closed under independent joins if is an abstract class, and if for every $L \in \mathcal{L}_{\mathcal{M}_c}$ and any independent subset A of L, with $(a/0_L) \in \mathcal{C}$, $\forall a \in A$, one has that

$$\left(\bigvee A/0_L\right) \in \mathcal{C}.$$

Definition

We say that a class $\mathcal{C}$ in $\mathcal{L}_{\mathcal{M}_c}$ is natural if the following holds:

(i) *$\mathcal{C}$ is hereditary;*

(ii) *$\mathcal{C}$ is closed under essential extensions;*

(iii) *$\mathcal{C}$ is closed under independent joins.*

As targeted

Proposition

Let $\mathcal{C}$ be a hereditary class in $\mathcal{L}_{\mathcal{M}_c}$. Then, $\mathcal{C}^{\perp_\leq}$ is a natural class.

Proposition

Let $\mathcal{C}$ be a natural class in $\mathcal{L}_{\mathcal{M}_c}$. Then,

$$\mathcal{C} = (\mathcal{C}^{\perp_\leq})^{\perp_\leq}.$$

The last two propositions yield to the next:

Theorem

For $\mathcal{L}_{\mathcal{M}_c}$, the skeleton of the big lattice of hereditary classes of linear lattices is precisely the collection of all natural classes of linear lattices.

The big lattice $\mathcal{L}_{nat}$

We denote by $\mathcal{L}_{nat}$ the skeleton of $\mathcal{L}_{her}$.

Remark

- *$\mathcal{L}_{nat}$ is ordered by inclusion.*
- *$\mathcal{L}_{nat}$ is closed under intersections. Moreover, one can see that, for any family $\{\mathcal{N}_i\}_{i\in I}$ of natural classes, one has that*

$$\left(\bigcup_{i\in I}\mathcal{N}_i\right)^{\perp_{\leq}\perp_{\leq}}$$

is the least upper bound in $\mathcal{L}_{nat}$ for $\{\mathcal{N}_i\}_{i\in I}$.

- *Hence, $\mathcal{L}_{nat}$ defines a big lattice.*

Just as in R-Mod, complements in $\mathcal{L}_{nat}$ are unique:

Lemma

Let $\mathcal{N} \in \mathcal{L}_{nat}$. If $\mathcal{D}$ is a complement for $\mathcal{N}$ in $\mathcal{L}_{nat}$, then, $\mathcal{D} = \mathcal{N}^{\perp_{\leq}}$.

With this in mind, as complements in $\mathcal{L}_{nat}$ are unique, and these are also strong pseudocomplements, [6, Chapter III, Prop 4.4] implies that

Theorem

$\mathcal{L}_{nat}$ is a big Boolean lattice.

Some other results in $\mathcal{L}_{\mathcal{M}}$

Some other results are:

- The big lattice of cohereditary classes of linear lattices. In this case, its skeleton consists of the big lattice of conatural classes of linear lattices;
- The big lattice of linear lattices preradicals;
- Torsion theories and open classes of linear lattices;
- Semi-projective linear lattices;
- Semi-injective linear lattices.

THANK YOU!

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